

## THE TREATMENT OF SERIES IN THE *GAṆITASĀRASAMĠRAHA* OF MAHĀVĪRĀCĀRYA AND ITS CONNECTIONS TO JAINA COSMOLOGY

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### 1. Introduction

It is easy to understand the enthusiasm shown by the well-known historian of mathematics, David Eugene Smith, in his speech at the Fourth International Congress of Mathematicians (Rome 1908) announcing the forthcoming publication of a new and significant Sanskrit text on mathematics edited, and translated into English, by a scholar from South India, Professor Malur Rangacharya.<sup>1</sup> This text was the *Gaṇitasārasamᠠgraha* (GSS), composed by Mahāvīrācārya, a Digambara Jaina *ācārya*, probably attached in some way to the court of King Amoghavarṣa Nṛpatuṅga (ca. 814 - 878) of the Rashtrakuta dynasty. The historical value of its contents was of no doubt to either M. Rangacharya or D. E. Smith, as the Sanskrit works known to the Western world up to the beginning of the 20<sup>th</sup> century were mainly the classics authored by Āryabhaṭa (end of 5<sup>th</sup> c.), Varāhamihira (6<sup>th</sup> c.), Brahmagupta (7<sup>th</sup> c.) and Bhāskarācārya (12<sup>th</sup> c.), most of these works having been translated in the first part of the 19<sup>th</sup> century by British Indologists (particularly H. T. Colebrooke). According to D. E. Smith, who was planning to write a general *History of Mathematics*, the study of the GSS would then shed “new light upon the subject of Oriental mathematics, as known in another part of India and at a time about midway between that of Aryabhata and Bhaskara, and two centuries later than Brahmagupta” (Smith 1908: 106).

The long-awaited publication was finally released in 1912 at the University of Madras, where M. Rangacharya had been appointed in 1901 as Professor of Sanskrit and Comparative Philosophy, as well as Curator of the Library of Oriental Manuscripts (GOML). Even before the recognition he received for this work on the GSS, he had already been awarded the prestigious “Rao Bahadur” medal and title in 1903, in recognition of his profound scholarship.<sup>2</sup>

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<sup>1</sup> In his speech, D. E. Smith first complained that native scholars in India were doing too little to bring to light the ancient material known to exist and to make it known to the Western world. In his opinion, this neglect was due to the fact that “it is hard to find a man with the requisite scholarship, who can afford to give his time to what is necessarily a labor of love,” rather than a lack of existing Sanskrit manuscripts (Smith 1908: 106).

<sup>2</sup> This medal and title (lit. Most honourable Prince) was a great honour bestowed upon individuals, during British rule, for their service to the Empire. For more information on M. Rangacharya, see Gupta 2013.

Subsequently, his 1912 publication became a landmark in the historiography of mathematics in India, as the GSS contains more than a thousand versified stanzas, and offers a large and unprecedented choice of algorithmic prescriptions and details on many different mathematical topics.<sup>3</sup>

Much has been written since 1912 about the GSS's contents and some of its verses have become classics. We can mention, for example, the frequently quoted eulogy on the science of calculation (*gaṇita*), acclaimed as applicable to all kinds of fields (GSS 1.9-16), the elaborate metaphor linking different mathematical topics to different parts of an ocean (GSS 1.20-23) and Mahāvīra's characteristic style of poetry embedded in his problem statements.<sup>4</sup>

In their groundbreaking historical survey *History of Hindu Mathematics*, first published in Lahore in 1935, Datta and Singh drew heavily from the GSS. They compared many of the computational algorithms proposed by Mahāvīrācārya to those of other authors, to highlight points of similarity or differences. However, their goal was primarily to convey the idea that India's mathematical past had been strong, coherent and fairly uniform, rather than to analyze the peculiarities found in the GSS.

Some patterns in Sanskrit mathematical works began to emerge. For example, texts could be mere chapters of larger astronomical compositions or could be independent manuscripts, often called *pāṭī* or *pāṭīgaṇita* (board mathematics) works. In this case, they consisted of two separate main sections presented in an ordered way: first the "operations" (*parikarman*) and then the "practices" or "procedures" (*vyavahāra*).<sup>5</sup> A list of the topics constituting a body of mathematical knowledge had already been hinted at by Brahmagupta (7<sup>th</sup>

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<sup>3</sup> It was translated into Hindi in 1963 by L. C. Jain, more recently into Kannada in 2010 by Padmavathamma and finally reprinted in 2011 after decades of unavailability.

<sup>4</sup> One of the most famous examples is the elaborate and poetic description of a forest which is provided as an introduction to a series of problems dealing with the *kuṭṭīkāra* (GSS 6.116½ - tr. Rangacharya): "Into the bright and refreshing outskirts of a forest, which were full of numerous trees with their branches bent down with the weight of flowers and fruits, trees such as *jambū* trees, lime trees, plantains, areca palms, jack trees, date-palms, *hintāla* trees, palmyras, *punnāga* trees, and mango trees - (into the outskirts), the various quarters whereof were filled with the many sounds of crowds of parrots and cuckoos found near springs containing lotuses with bees roaming about them - (into such forest outskirts) a number of weary travellers entered with joy." The numerical data starts in the following stanza (GSS 6. 117½ - tr. Rangacharya): "(There were) 63 (numerically equal) heaps of plantain fruits put together combined with 7 (more) of those same fruits; and these were (equally) distributed among 23 travellers so as to leave no remainder. You tell (me now) the (numerical) measure of a heap (of plantains)."

<sup>5</sup> The Sanskrit words *parikarman* and *vyavahāra* used to be translated as "logistics" and "determinations" but now the terms "operations" and "procedures" or "practices" seem to be preferred.

c.).<sup>6</sup> It was detailed in the ninth century by his commentator Pṛthūdakasvāmin, who stated that the twenty logistics were the eight arithmetical operations (addition, subtraction, multiplication, division, square, square-root, cube, cube-root), the five rules of reduction relating to the five standard forms of fractions, the six rules relating to proportions (rule of three, inverse rule of three, the rules of five, seven, nine and eleven), and barter and exchange; the eight determinations were the treatment of mixtures, progressions or series, plane figures, excavations, stock, saw, mound and shadow (Datta & Singh 1935: 124).

### Some Unanswered Questions

Much research work, new studies and translations have been undertaken since then, and recent works seem to indicate that the earlier idea of uniformity may have to be challenged.<sup>7</sup> More importantly, as K. Plofker (2009a: 296) states at the end of her well-documented synthesis *Mathematics in India*, they show that there are still some very basic issues about the Indian mathematical tradition, which are incompletely understood. In her view, the first of the fundamental issues that remains largely unanswered concerns classification and structure, and she turns to the case of the GSS to illustrate her point:

What determined the basic building blocks of subjects of mathematics? For example, why did Mahāvīra consider it possible in the *Gaṇita-sāra-saṅgraha* to dispense with addition and subtraction of numbers as canonical arithmetic operations? How did the operations and procedures of medieval arithmetic texts originate, and how did a particular problem get assigned to a particular category?

Indeed, a quick glance through the titles and sub-titles M. Rangacharya includes in his edition indicates that the GSS's overall structure does not conform to what has been considered the norm for *pāṭī* works.<sup>8</sup> The first issue concerns the separation that is supposed to exist between the *parikarman* and *vyavahāra* sections: this is not respected here since the *parikarman* is itself

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<sup>6</sup> “He who distinctly and severally knows the twenty logistics, addition, etc., and the eight determinations including (measurement by) shadow is a mathematician” (Datta & Singh 1935: 124).

<sup>7</sup> For example, in an article written with A. Keller, we have compared several methods of multiplication indicated by different authors and demonstrated that the claim of similarity made by Datta and Singh was highly disputable. See Keller & Morice-Singh, forthcoming.

<sup>8</sup> These titles and sub-titles do not appear in the manuscripts: it was M. Rangacharya's editorial choice to insert them in his edition.

the topic of the first *vyavahāra*, Mahāvīrācārya having divided his entire work into only eight *vyavahāras*. Furthermore, his list of *vyavahāras* in the GSS cannot be completely identified with the one suggested by Pṛthūdakasvāmin, and no other author has retained such a model.

The second issue concerns the detailed contents of the *parikarman* topic: authors usually start with arithmetical operations on integers, sometimes repeat the same for fractions and also add more features. For example, Bhāskaraācārya considers the “three-quantity-operation” (*trairāsika*) as a part of the *parikarman*, while Mahāvīrācārya does not. In consequence, the total number of “fundamental operations” varies from 16 to 39 as one can see from Table 1 below (SaKHYa 2009: xl).<sup>9</sup>

Works	BSS	PG	Tr	GSS	MS	SS	GT	SGT	L	GSK	GK
Number of <i>parikarmāṇi</i>	20	29	(31)	(16)	(39)	20	(20)	(36)	(24)	25	(24)
Arithmetical Operations											
Integers											
addition	(1)	1*	1*	7*	1	(1)	1	1	1	1*	1
Subtraction	(2)	2*	2*	8*	2	(2)	2	2	2	2*	2
Multiplication	(3)	3	3	1	3	3	3	3	3	3	3
Division	(4)	4	4	2	4	4	4	4	4	4	4
Square	(5)	5	5	3	5	5	5	5	5	5	5
Square root	(6)	6	6	4	7	7	6	6	6	6	6
Cube	7	7	7	5	6	6	7	7	7	7	7
Cube root	8	8	8	6	8	8	8	8	8	8	8
Fractions											
addition	1	9	9	15*	21	1	1	9	9	9	9
Subtraction	2	10	10	16*	22	2	2	10	10	10	10
Multiplication	3	11	11	9	23	3	3	11	11	11	11
Division	4	12	12	10	24	5	4	12	12	12	12
Square	5	13	13	11	25	4	5	13	13	13	13
Square root	6	14	14	12	26	(6)	6	14	14	14	14
Cube	(7)	15	15	13	27	(7)	7	15	15	15	15
Cube root	(8)	16	16	14	28	(8)	8	16	16	16	16

(continued)

Table 1: Fundamental Operations

The set of the eight canonical operations (*parikarmāṣṭaka*) could be expected to be stable for most authors but this again does not hold true for Mahāvīrācārya, who is the only one to start with multiplication and to still deal with eight operations in total. The other author who begins with multiplication is Śrīpati (11<sup>th</sup> c.), in his *Siddhāntaśekhara* (SS). However, the situation

<sup>9</sup> Whenever a number is within brackets, it means the subject has not been specified or treated explicitly by the author. The works indicated here are the *Brāhmasphuṭasiddhānta* (BSS), *Pāṭiganīta* (PG), *Triśatikā* (Tr), *Gaṇitasārasaṅgraha* (GSS), *Mahāsiddhānta* (MS), *Siddhāntaśekhara* (SS), *Gaṇitatilaka* (GT), *Siṃhatilaka*’s commentary on the GT (SGT), *Līlāvātī* (L), *Gaṇitasārakaumudī* (GSK) and *Gaṇitakaumudī* (GK).

here is different: he starts with multiplication because he does not deal with elementary addition and subtraction. He then expounds only the six remaining operations for integers.

One can see from Table 1 above that Mahāvīrācārya has replaced the first two operations (ordinary addition and subtraction of integers) by a seventh and an eight operation (noted 7\* and 8\*), which correspond respectively to the “sum” (*saṃkalita*) of an arithmetical or a geometrical progression, and the “difference” (*vyutkalita*) between two of those sums.<sup>10</sup>

The calculations involved in these last two operations require a knowledge of the previous operations, and this explains their position as the last ones in the list. The arrangement is the same for the operations involving fractions, hence, operations 15\* and 16\* deal with series having fractional parameters.

An extended treatment of series within the section on arithmetical operations is unexpected in a *pāṭī* work, as it should find its place in a specific *vyavahāra*, called *śreḍhī-vyavahāra* (lit. practice on series), and not in the *parikarman* section. According to K. Plofker (2009a: 163), this shows that Mahāvīrācārya has deliberately “cast out” the elementary operations of addition and subtraction from the classificatory structure of mathematics, and she finds this “quite daring.”

I would like to add another issue to those described above: the treatment of the last two operations occupies an overwhelming place in the chapter, nearly half of it, as shown here:<sup>11</sup>

60 stanzas	55 stanzas
multiplication ( <i>guṇakāra, pratyutpanna</i> )	addition ( <i>citi, saṃkalita</i> )
division ( <i>bhāgahāra</i> )	32 → arithmetical progressions (22 rules, 10 sample problems)
square ( <i>kṛti</i> )	13 → geometrical progressions (7 rules, 6 sample problems)
square root ( <i>varga-mūla</i> )	subtraction ( <i>vyutkalita, śeṣa</i> )
cube ( <i>ghana</i> )	10 → for both cases (5 rules, 5 sample problems)
cube root ( <i>ghana-mūla</i> )	

Table 2: Contents of the first *vyavahāra*

I intend to show in this paper that some answers to these questions can be found by an exploration of the mathematical content of Jain cosmological texts, but, before that, I will present some characteristic features of the treatment of series in the GSS itself.

<sup>10</sup> “Progressions” or “sequences” are often translated as “series” (*śreḍhī*), since the aim is often to calculate the sums of their terms. A star beside the serial numbers 1 and 2, means the series considered is the natural series. In other words,  $S = 1 + 2 + \dots + n$ .

<sup>11</sup> We don’t know the reason for such an importance given to these two operations.

## 2. The Treatment of Series in the GSS

Methods of summation for arithmetical progressions were well known in India centuries before Mahāvīracārya. In the *Āryabhaṭīya* (ca. 500), for example, Āryabhaṭa provides a very concise verse (Ab 2.19) in which, according to his commentator Bhāskara I (7<sup>th</sup> c.), five rules can be read.<sup>12</sup> In brief and with modern notation, if  $(u_n)$  is an arithmetical progression having a first term noted  $u_1$  and a common difference  $r$ , the *sūtra* Ab 2.19 indicates how to calculate the sum  $(u_{p+1} + \dots + u_{n-1} + u_n)$  of the last  $n - p$  terms, the value of  $n - p$  being chosen.<sup>13</sup>

If  $p = 0$ , the total sum  $(u_1 + u_2 + \dots + u_p + u_{p+1} + \dots + u_{n-1} + u_n)$  is obtained. In other words, this *sūtra* provides an algorithm to calculate a sum starting from any term, up to the last.

### Two “New” Operations

Mahāvīracārya’s computational techniques to deal with arithmetical progressions are rooted in the same mathematical knowledge as Āryabhaṭa’s, but they are greatly amplified and presented differently. He also deals with geometrical progressions in great detail, but most importantly, he innovates by creating two distinct operations with respect to series. The first one, called “addition” (*saṃkalita*, lit. made together), corresponds to the summing of successive terms of a sequence starting from its first term. The second operation, called “subtraction” (*vyutkalita*, lit. made apart), consists in computing a “remainder-series” which is equal to the difference between the sum of the entire series and the sum of a chosen first part (*sveṣṭa-vitta*), as the number of terms in the first part is chosen. In modern notation we can write:

addition:  $(u_1 + u_2 + \dots + u_p) + (u_{p+1} + \dots + u_n) = (u_1 + u_2 + \dots + u_n) = \text{total sum}$

subtraction:  $(u_1 + u_2 + \dots + u_n) - (u_1 + \dots + u_p) = \text{difference}$   
 $\text{total sum} - \text{chosen first part} = \text{difference}$

We obviously have:  $(u_1 + u_2 + \dots + u_n) - (u_1 + \dots + u_p) = u_{p+1} + u_{p+2} + \dots + u_n$ .

The value of the difference obtained by Mahāvīracārya is then identical to the result that can be calculated directly using Āryabhaṭa’s *sūtra*. However, the authors have contrasting definitions

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<sup>12</sup> For complete details, see Keller 2006 2: 106-10.

<sup>13</sup> AB 2.19: *iṣṭaṃ vyekaṃ dalitaṃ sapūrvam uttara-guṇaṃ samukhaṃ madhyam / iṣṭa-guṇitaṃ iṣṭa-dhanaṃ tv athavādy-antaṃ padārdha-hatam //*

of what is “chosen”: for Āryabhaṭa it is the number of terms in the last part and for Mahāvīrācārya, the number of terms in the first part that is subtracted from the total.

From Table 2 above, it can also be seen that the 8<sup>th</sup> operation in the GSS is far less developed than the 7<sup>th</sup> (10 stanzas / 45 stanzas). We can infer that the operation of subtraction may have been created for the sake of reciprocity, as operations are known to go in pairs like multiplication/division, square/square root, etc., and to ensure a total of eight operations.

### Computing Sums (*saṃkalita*)

The first four *sūtras* (GSS 2.61-64) dealing with “addition” (*saṃkalita*) provide algorithms to compute the sum of a sequence of terms in arithmetical progression.<sup>14</sup> In the GSS, the technical vocabulary attached to this subject is standard. The “first term” in modern mathematics is replaced by any word that suggests “beginning,” “origin,” “opening,” like *ādi*, *prabhava*, *mukha*, *vadana*, etc., and for “common difference,” words meaning “increase,” “increment,” “excess” or “difference” etc. are used, mainly *caya*, *pracaya*, and *uttara*. The “number of terms” is generally called *pada* or *gaccha* (step, stride, period, etc.) and the total sum is named *sarva-dhana* (value of the total) or sometimes *saṃkalita* (sum) or even *gaṇita* (calculation).

The presentation of the numerical data for sample-problems (*uddeśaka*) relating to series appears to have always been the same in the manuscripts. As the illustrations below show, the numerical values relating to the ten series, whose sums are asked for in the first sample-problem (GSS 2.65), are shown in a table containing three lines always presented in the same order. One can read “*ga*” for *gaccha*, “*u*” for *uttara* and “*ā*” for *ādi* in the first column.

ग	५	७	९	११	१३	१५	१७	१९	२१	सर्वेषु
उ	१	२	३	४	५	६	७	८	९	५५
आ	१	२	३	४	५	६	७	८	९	५५

Fig 1: AS Bombay 229. From Bhāu Dājī, fol. 12, v°

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<sup>14</sup> For example, to compute the sum of the terms 5, 7, 9, 11, 13, 15, 17, 19, 21. The first term  $u_1$  is 5, the common difference  $r$  is 2 and the number of terms  $n$  is 9. Hence, the sum  $S$  will be:  $S = u_1 + u_2 + \dots + u_9 = 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21$ . Different formulas can be applied to compute it.

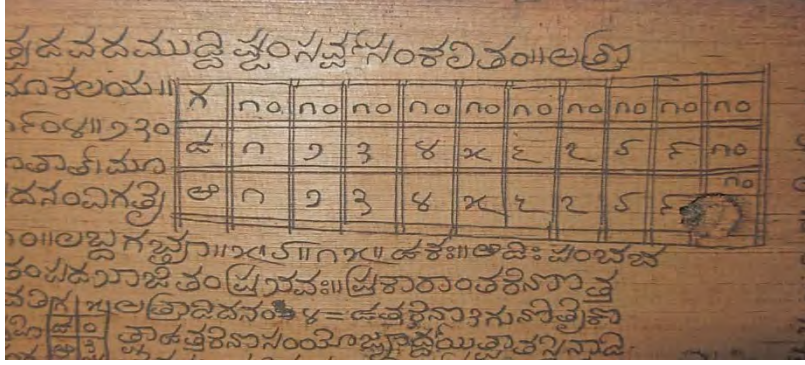


Fig 2: GOML-13409, fol. 5, v°

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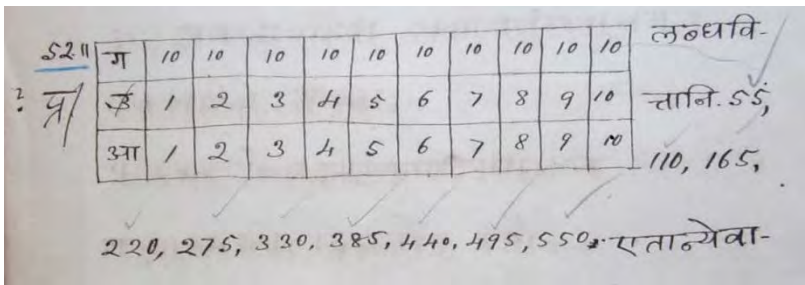


Fig 3: GOML-13412, extract from the running commentary p. 42

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The third illustration above is an extract of the paper manuscript GOML-13412 that provides the transcription in Nagari script of a text probably originally in Kannada characters. It seems to have been used by M. Rangacharya as a kind of “pre-print” copy for his publication.<sup>16</sup>

In the case of this first example, GSS 2.65, it can be observed that he attempted to replace the “u” for *uttara*, in the table with a “pra” instead, as, in this problem statement, the “increase” is called *pracaya* and not *uttara*.

To find the sum of  $(u_1 + u_2 + \dots + u_n)$  nowadays, we would presumably calculate the “last term” (*antya-dhana*) if unknown, by applying the formula  $u_n = u_1 + (n - 1)r$ , then the “mean term” (*madhya-dhana*) equal to  $\frac{u_1 + u_n}{2}$ , and multiply this by the number of terms. This corresponds to what is prescribed in GSS 2.64. In the 9<sup>th</sup> century, however, algebraic formulations were not in use and computations were carried out step by step, following prescriptions set out in an algorithm, for instance the *sūtras* GSS 2.61 or GSS 2.62. Both procedures give exact answers, but are distinct:<sup>17</sup>

<sup>15</sup> In this paper, all the photos of manuscripts from the Government Oriental Manuscripts Library are the author’s.

<sup>16</sup> See Morice-Singh 2015: 28f.

<sup>17</sup> In this paper, all the translations are the author’s, unless otherwise specified.



GSS 2.61: The number of terms (*gaccha*) diminished by *one* (*rūpa*) is halved [and] multiplied by the increase (*pracaya*); [this] combined with the first term (*prabhava*) [and then] multiplied by the number of terms (*pada*) becomes the sum (*saṅkalita*) of all [the terms].<sup>18</sup>

GSS 2.62: The number of terms (*gaccha*) diminished by *one* (*rūpa*) is multiplied by the increase (*pracaya*), [this is] combined with twice the first term (*ādi*); [the sum] multiplied by the number of terms (*gaccha*), divided by *two*, will become in all cases the sum (*saṅkalita*).<sup>19</sup>

Using modern notation we have:

<p>GSS 2.61</p> $n - 1$ $(n - 1)/2$ $[(n - 1)/2] \times r$ $[(n - 1)/2] \times r + u_1$ $\{[(n - 1)/2] \times r + u_1\} \times n$ $\left( \frac{n-1}{2} \times r + u_1 \right) \times n \rightarrow S$	<p>GSS 2.62</p> $n - 1$ $(n - 1) \times r$ $(n - 1) \times r + 2u_1$ $[(n - 1) \times r + 2u_1] \times n$ $\{[(n - 1) \times r + 2u_1] \times n\} / 2$ $\frac{[(n-1)r + 2u_1] n}{2} \rightarrow S$
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Another efficient way of calculating a sum, which seems to have been popular in the Jain scholarly tradition, is to break up the total into two parts called *ādi-dhana* and *uttara-dhana*, respectively.<sup>20</sup> Indeed, the sum *S* can be written in different ways:

$$S = u_1 + (u_1 + r) + (u_1 + 2r) + \dots + (u_1 + (n - 1)r)$$

$$S = (u_1 + u_1 + \dots + u_1) + (r + 2r + \dots + (n - 1)r)$$

$$S = (u_1 + u_1 + \dots + u_1) + r(1 + 2 + \dots + (n - 1)).$$

<sup>18</sup> GSS 2.61: *rūpeṇono gaccho dalī-kṛtaḥ pracaya-tāḍito miśraḥ / prabhaveṇa padābhyastaḥ saṅkalitam bhavati sarveṣāṃ //*

<sup>19</sup> GSS 2.62: *eka-vihīno gacchaḥ pracaya-guṇo dvi-guṇitādi-samyuktah / gacchābhyasto dvi-kṛtaḥ prabhavet sarvatra saṅkalitam //*

<sup>20</sup> This is done in some prose passages of the TP and in the *Trilokasāra* (TLS), for instance, for the computation of the number of moons above the successive continents and seas.

The first part ( $u_1 + u_1 + \dots + u_1$ ) is the *ādi-dhana* or the “[cumulative] value of the first term.” It is equal to  $u_1 \times n$ . The second part  $r(1 + 2 + \dots + (n - 1))$  is the *uttara-dhana* or the “[cumulative] value of the increase” and is equal to  $n \times r \times \frac{n-1}{2}$ .<sup>21</sup>

As stated in GSS 2.63, we have: *ādi-dhana* + *uttara-dhana* = *sarva-dhana*.<sup>22</sup>

An interesting and unusual remark ends the *sūtra* GSS 2.63: “the increment is subtractive when the last term is made to be the first” (*ūnottaram mukhe ’ntya-dhane*). For instance, the sequence 5, 7, 9, 11, 13, 15, 17, 19, 21 can be reversed as 21, 19, 17, 15, 13, 11, 9, 7, 5. The sum does not change but the increase here becomes “subtractive,” with  $r$  equal to  $-2$ . No other author apart from Mahāvīrācārya makes this kind of remark, and we would like to understand what could have incited him to do so.

### Some Examples from Manuscripts

Let us see how the calculations were carried out taking, for example, the problem GSS 2.66:

GSS 2.66: A certain excellent *śrāvaka* gave gems in offering to five temples (one after another) commencing (the offering) with two (gems), and then increasing (it successively) by three (gems). O you, who know how to calculate, mention what their (total) number is (Tr. Rangacharya).

From the running commentary p. 44-45 in GOML-13412 we find:<sup>23</sup>

<i>ga</i>	5
<i>u</i>	3
<i>ā</i>	2

Setting down: the beginning (*mukha*) multiplied by the number of terms (*pada*), one gets 10; this is precisely the *ādi-dhana*; by halving the number of terms decreased by one (*eka-rahita-pada*), we get 2. The increase (*pracaya*) is multiplied, we get 6. The number of terms multiplied by this

10
30

is 30. This is precisely the *uttara-dhana*. The sum of the *ādi(-dhana)* and *uttara-dhana* is 40. This is precisely the total sum (*sarva-dhana*).

<sup>21</sup> The computation of sums of series of natural numbers was well known in India since 500 B.C. at least (Datta & Singh 1993: 104). The formula is:  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ .

<sup>22</sup> There seems to be some confusion about the meaning of *ādi-dhana*. Datta and Singh understand it as the “value of the first term,” *i.e.* the value of  $u_1$  only, while it is obviously different in the GSS (Datta & Singh 1993: 105). Also, specific names for *ādi-dhana* and *uttara-dhana* do not exist in our mathematics today.

<sup>23</sup> *nyāsaḥ mukham pada-guṇitaṃ jātaṃ 10 etad evādi-dhanam / eka-rahita-padārdhenānena 2 pracayo guṇito jātaḥ 6 anena gacchaḥ guṇitaḥ 30 idam eva uttara-dhanam ādy-uttara-dhana-yogaḥ 40 etad eva sarva-dhanam /*

It can be presumed that the fixed and regular presentation of the data, always on three lines in the same order, was a great visual support to help memorize the algorithm, as the computations to be done would then be translated as movements or actions on these three lines, and the execution of the algorithm could hence become mechanical.

### Computing Other Elements

Many more rules are set out in the GSS and it is not possible to do justice to all of them in this paper. The goal was generally to compute  $S$  or any of the three quantities  $u_1$ ,  $r$ ,  $n$ , when  $S$  is known.<sup>24</sup>

For example, to calculate  $n$ , the number of terms (*gaccha*), in case  $u_1$ ,  $r$  and  $S$  are known, we can apply the algorithms GSS 2.69 or GSS 2.70, equivalent respectively to the formulas

$$n = \left[ \frac{\sqrt{S \times 8 \times r + (2u_1 - r)^2} + r}{2} - u_1 \right] \div r \text{ and } n = \left[ \frac{\sqrt{S \times 8 \times r + (2u_1 - r)^2} - (2u_1 - r)}{2} \right] \div r.$$

The first one is used in the running commentary included in the GOML-13409 to solve the problem GSS 2.71. The number of terms (*gaccha*) being unknown, it is indicated by a “0” (zero) in the table.

GSS 2.71: The first (*ādi*) is two, the increase (*pracaya*) is eight. These two are increased successively by one (*rūpa*), till three [series are so made up]. The sums of the three series are respectively zero(*kha*)-nine(*aṅka*), six(*rasa*)-sept(*adri*)-two(*netra*) and zero(*kha*)-one(*indu*)-eleven(*hara*). What is the number of terms in each series?<sup>25</sup>

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<sup>24</sup> The text raises other questions that we do not usually encounter in our mathematics of today. For example, if  $S$  is known, can the other elements be identified in order to obtain a new sum which will be double, triple, etc. or half, third, etc. of the initial sum? Or if the value of  $S + u_1$  is known, can it be split in order to provide suitable values of  $S$  and of  $u_1$  separately? See Morice-Singh 2015: 239-48.

<sup>25</sup> GSS 2.71: *ādi-dvau pracayo'ṣṭau dvau rūpeṇa trayāt kramād vṛddhau / khāṅkau rasādrinetraṃ khenduharā vittam atra ko gacchaḥ* // Here the numbers are expressed by stating the digits (one or two at a time) starting with the digit in the units' position. For example, zero (*kha*) - nine (*aṅka*) is equal to 90, etc. This popular system of expression of numbers is called the *bhūta-saṃkhya* system. In the GSS, Mahāvīrācārya made ample use of it, often drawing names from the Jain terminology itself, like *leśyā*, for example, which was incorrectly interpreted by M. Rangacharya as “*lekhya*.” See Morice-Singh 2015: 63-74, 2016: 42 and Sarma 2009 for more details on the expression of numbers.

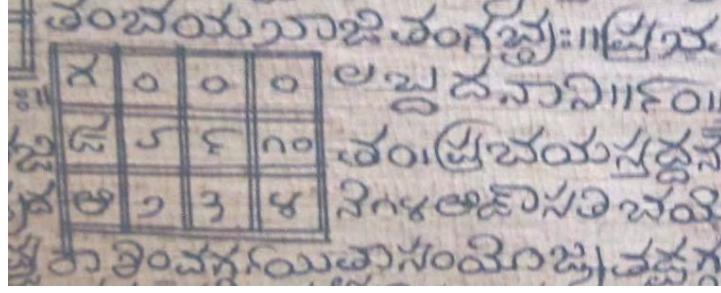


Fig 4: GOML-13409, fol. 5, v<sup>o</sup>

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GOML-13412, extract from the running commentary p. 47, for the first question:

<i>ga</i>	0	0	0
<i>u</i>	8	9	10
<i>ā</i>	2	3	4

Sums obtained 90, 276, 1110. Number of terms obtained 5, 8, 15.

They are brought like this, for instance: The quantity (*rāsi*) is 90; multiplied by 8, it becomes 720. Multiplied by the increase (*uttara*), it is 5760. The beginning (*ādi*) multiplied by 2 is 4, the increase is 8, the

difference is 4. Its square (*kṛti*) is 16. Together, 5776. Now the (square-) root becomes 76. Along with the increase (*caya*), 84. Its half, 42. The beginning (*ādi*) removed, 40, divided by the increase (*caya*), 5. This is the number of terms (*gaccha*) obtained.<sup>26</sup>

In other cases, it can be the first term or the increase that is asked for, like in GSS 2.77:

GSS 2.77: The first term (*vadana*) is nine, the number of terms (*pada*) seven (*tattva*), the sum (*dhana*) one hundred increased by five (*bhāva*); how much is the increase? The increase (*caya*) is five, the number of terms (*pada*) is eight, the sum (*dhana*) one hundred and fifty-six, say what is the first term (*mukha*).<sup>27</sup>

This example is solved in the running commentary of GOML-13412, p. 51:

<i>ga</i>	7	8
<i>u</i>	0	5
<i>ā</i>	9	0

The value of the sum (*saṅkalita-dhana*) is 105; by removing the *ādi-dhana* which is 63, one gets 42; the number of terms (*pada*) squared is 49, the number of terms (*pada*) removed, 42; by halving it, 21; after dividing, 2; this is precisely the increase (*pracaya*). The first term (*ādi*)

<sup>26</sup> *labdha-dhanāni* 90, 276, 1110. *labdha-gacchaḥ* 5, 8, 15 *etad evānīyante | tathā hi | rāsiḥ* 90 *aṣṭa-guṇito jātaḥ* 720 *uttareṇa guṇitaḥ* 5760 *dvi-guṇitādi* 4 *uttaram* 8 *anayor viśeṣaḥ* 4 *asya kṛtiḥ* 16 *anayā sahitaḥ* 5776 *ato'panīta-mūlam* 76 *caya-yutam* 84 *ardhitam* 42 *ādinā rahitam* 40 *caya-hṛtam* 5 *jāto gacchaḥ |*

<sup>27</sup> GSS 2.77: *nava-vadanaṃ tattva-padaṃ bhāvādhika-śata-dhanaṃ kiyān pracayaḥ | pañca cayo'ṣṭa padaṃ ṣaṭ-pañcāśac-chata-dhanaṃ mukhaṃ kathaya ||*

is brought, for instance, the total (*gaṇita*) is 156. By removing the *uttara-dhana* which is 140, remains 16, having divided by the number of terms (*pada*), one has 2, this is precisely the first term (*prabhava*).<sup>28</sup>

All these results are based on the algorithms given in GSS 2.73 which are equivalent to the formulas  $\frac{S-u_1n}{(n^2-n)/2} = r$  and  $\frac{S-n \times r \times \frac{n-1}{2}}{n} = u_1$ , and are not difficult to justify.

In many cases, Mahāvīrācārya proposes several algorithms to arrive at the same result, giving the calculator some flexibility, as he can then choose the easiest or quickest method, if desired. For instance, here, the increase and the first term could be obtained with the help of three algorithms (GSS 2.73-74-75): one may choose one of them depending on whether  $S$  is divisible or not by  $n$ , if  $n$  is odd or even, etc.

### 3. Geometrical Progressions in the GSS

The shift from arithmetical to geometrical progressions is not clearly indicated in the manuscripts. It has to be deduced from the appearance of a new terminology: the increase (*uttara*) becomes the common ratio  $q$ , which is called *guṇa* or *guṇottara* (multiplicative increase), but sometimes only *uttara*. The names for the first term and the number of terms do not change.<sup>29</sup> The last term (*antya-dhana*) is defined as it is today ( $u_n = u_1 q^{n-1}$ ) and there is something new, the *guṇa-dhana* ( $u_1 q^n$ ), equivalent to the term that comes after the *antya-dhana*. This does not have a specific name in our modern mathematics. The sum is called *guṇa-saṅkalita* or “multiplicative summation.”

The first task is to obtain the value of the sum:  $S = \frac{u_1 q^n - u_1}{q-1}$  or  $S = \frac{(q^n - 1) \times u_1}{q-1}$ . The first formula corresponds to the statement in GSS 2.93: “after removing the first term from the *guṇa-dhana*, one divides by the common ratio minus *one*.”

---

<sup>28</sup> *saṅkalita-dhanam* 105 *ādi-dhanenānena* 63 *rahitam jātam* 42 *pada-kṛtiḥ* 49 *pada-rahitā* 42 *asya daḷena* 21 *sambhājītam* 2 *ayam eva pracayaḥ / ādir ānīyate / tathā hi / gaṇitam* 156 *uttara-dhanenānena* 140 *rahitam* 16 *pada-bhājītam* 2 *ayam eva prabhavaḥ /*

<sup>29</sup> For the sequence 5, 10, 20, 40, 80, 160, 320, 640, 1280, the first term  $u_1$  is 5, the common ratio  $q$  is 2 and the number of terms  $n$  is 9.

## A Powerful Algorithm

The challenge here can be to compute  $q^n$  as it can have a huge value in some cases. Mahāvīracārya then proposes a technique very similar to a prosody rule stated by Piṅgala.<sup>30</sup>

GSS 2.94: The number of terms (*gaccha*) [is transformed by the rule] “even-half-odd-zero-one (*sama-dala-viṣama-kha-rūpa*), multiplied by the multiplier (*guṇa-guṇita*) multiplied as a square (*varga-tāḍita*)”; [The result] diminished by one (*rūpona*), multiplied by the first term (*prabhava-ghna*), divided by the multiplier lessened by one (*vyekottara-bhājita*) is the total.<sup>31</sup>

The running commentary in the manuscript GOML-13409 contains a beautiful illustration showing how to use this rule to solve the second question in the sample problem GSS 2.100:

GSS 2.100: What is the wealth owned by a merchant when the first term (*mukha*) is seven, the multiplier (*guṇa*) three, the number of terms (*gaccha*) the square of three; and also with three, five, fifteen for the first term (*prabhava*), the common ratio (*guṇottara*), the number of terms (*pada*)?<sup>32</sup>

As stated in the second question,  $u_1 = 3$ ,  $q = 5$  and  $n = 15$  and one has to compute:

$5^{15} \rightarrow 5^{15} - 1 \rightarrow (5^{15} - 1) \times 3 \rightarrow \frac{(5^{15} - 1) \times 3}{5 - 1} = 22888183593$ . The difficulty here is to arrive at the value of  $5^{15}$  which is equal to 30517578125.

---

<sup>30</sup> In modern notation, this rule serves to calculate  $2^n$ . See Bag & Sarma 2003: 130f.

<sup>31</sup> GSS 2.94: *sama-dala-viṣama-kha-rūpo guṇa-guṇito varga-tāḍito gacchaḥ / rūponaḥ prabhava-ghno vyekottara-bhājitaḥ sāram //*

<sup>32</sup>GSS 2.100: *sapta-mukha-tri-guṇa-caya-tri-varga-gacchasya kiṃ dhanam vaṇijaḥ / trika-pañcaka-pañcadaśa-prabhava-guṇottara-padasyāpi //*

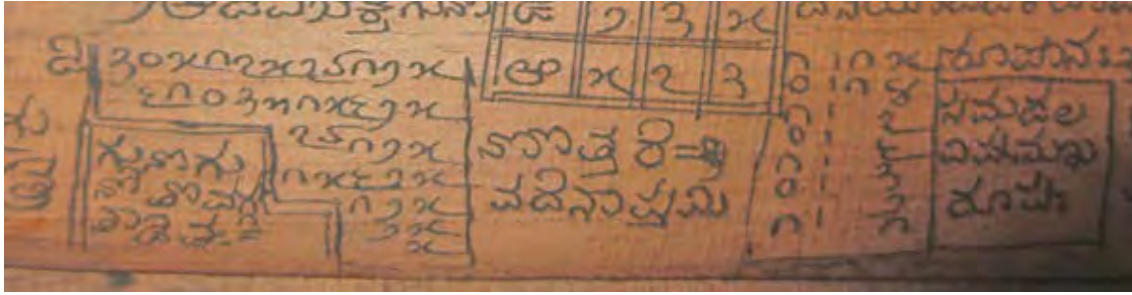
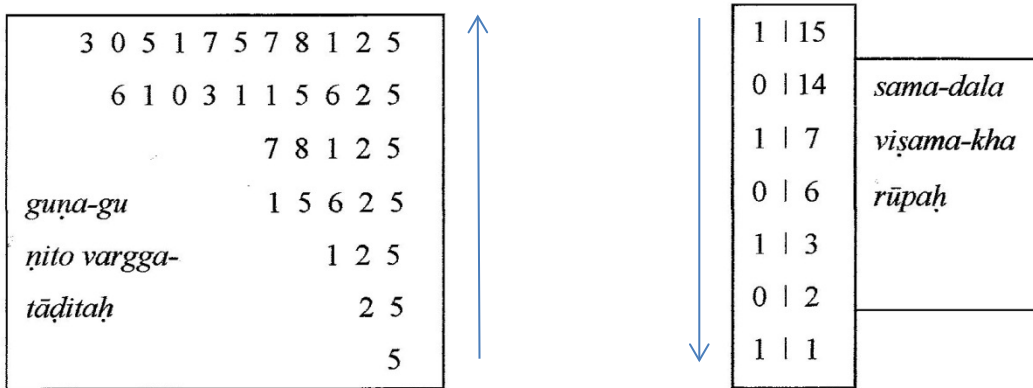


Fig 5: GSS-Goml-13409 fol. 7 r° (extract from the commentary, last lines at the bottom)

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One starts from the extreme right top, 15, and establishes the list of “ones” and “zeroes” according to the rule “*sama-dala-viṣama-kha-rūpa*,” depending on whether the number obtained is even or odd. Then, in the frame on the left, whenever faced with a “1” one multiplies by 5 and whenever faced with a “0” one calculates the square, moving from bottom to top.

The result for  $5^{15}$  is on the last line moving upwards, *i.e.* the top line. It is reached in a quicker manner than by calculating the product  $5 \times 5 \times 5 \times \dots \times 5$ , which requires fourteen multiplications.<sup>33</sup>

Many more *sūtras* on geometrical progressions are provided in this section. For example, there are descriptions of how to find a first term, a common ratio or a number of terms when the *guṇa-dhana* or the total sum is known.

<sup>33</sup> Śrīdhara provides the same kind of rule in his *Pāṭīgaṇita* (PG 94): *viṣame pade nireke guṇaṃ same’rdho kṛte kṛtiṃ nyasya / kramaśo rūpasyotkramaśo guṇa-kṛti-phalam ādinā guṇayet //PG-94//*. Bhāskarācārya (12<sup>th</sup> c.) does the same in the *Līlāvāṭī* (L.127). However, both authors suggest writing “*guṇa*” instead of “1” and “*kṛti*” instead of “0”, and their instructions are far more detailed than Mahāvīra’s, which seem to be in a style closer to Piṅgala’s writings. This remark may be an argument in favour of placing Śrīdhara chronologically after Mahāvīra.

#### 4. Computing Differences (*vyutkalita*)

As explained above, the operation of subtraction is meant to provide the value of the difference between an entire series ( $n$  terms) and a chosen first part of it ( $p$  terms):

$$(u_1 + u_2 + \dots + u_n) - (u_1 + \dots + u_p) = \text{difference.}$$

Each sum could be calculated separately and then their difference could be established, but, in the case of arithmetical progressions, an algorithm provides a more direct means of arriving at the result:

GSS 2.106: The chosen (*iṣṭa*) number of terms combined with the number of terms (*pada*), and the chosen number of terms are [both] lessened by one, divided by two, multiplied by the increase (*caya*), combined with the first term (*mukha*); these, when multiplied by the remaining (*śeṣa*) number of terms and by the chosen (*iṣṭa*) number of terms [give separately] the difference (*vyutkalita*) and the value of the chosen part (*sveṣṭa-vitta*).<sup>34</sup>

In modern notation:  $p + n \rightarrow p + n - 1 \rightarrow \frac{p+n-1}{2} \rightarrow \frac{p+n-1}{2} \times r \rightarrow \frac{p+n-1}{2} \times r + u_1 \rightarrow$   
 $\left(\frac{p+n-1}{2} \times r + u_1\right) \times (n - p) \rightarrow \text{vyutkalita}$   
 and  $p \rightarrow p - 1 \rightarrow \frac{p-1}{2} \rightarrow \frac{p-1}{2} \times r \rightarrow \frac{p-1}{2} \times r + u_1 \rightarrow \left(\frac{p-1}{2} \times r + u_1\right) \times p \rightarrow \text{sveṣṭa-vitta}$

The numerical data for the sample-problem GSS 2.111 is shown in the table below:

౧	౧౪	౩౨	౫౬	౮౬	౧౨౨	౧౬౪	౨౧౨	౨౬౬	౩౨౬	౩౯౨
౬	౩	౩	౩	౩	౩	౩	౩	౩	౩	౩
౪	౧	౧	౧	౧	౧	౧	౦	౦	౦	౦

Fig 6: GSS-Goml-13409, fol. 7, v°.<sup>35</sup> © Government Oriental Manuscripts Library

<sup>34</sup> GSS 2. 106: *sapadeṣṭaṃ sveṣṭaṃ api vyekaṃ dalitaṃ cayāhataṃ samukham / śeṣeṣṭa-gaccha-guṇitaṃ vyutkalitaṃ sveṣṭa- vittaṃ ca //*

<sup>35</sup> GSS 2.111: *dvi-mukhas tri-cayo gacchaś catur-daśa svepsitaṃ padaṃ sapta / aṣṭa-nava-ṣaṭka-pāñca ca kiṃ vyutkalitaṃ samākalaya // vyutkalita-dhanāni || 224 || 201 || 175 || 244 || 261 ||.*



<i>ga</i>	14	<i>i</i> 7	<i>i</i> 8	<i>i</i> 9	<i>i</i> 6	<i>i</i> 5	<i>śe</i> 7	<i>śe</i> 6	<i>śe</i> 5	<i>śe</i> 8	<i>śe</i> 9
<i>u</i>	3	3	3	3	3	3	3	3	3	3	3
<i>ā</i>	2	2	2	2	2	2	0	0	0	0	0

The total number of terms is 14 for the five cases; the chosen number of terms is indicated with an “*i*” (*iṣṭa*) and is successively 7, 8, 9, 6 and 5. The remaining number of terms is indicated by a “*śe*” (*śeṣa*) and is then obviously equal to 7, 6, 5, 8, and 9. The first term of a “remainder-series” being unknown, we find as usual a “0” in the table.

The computation of the “difference” for the second case (*i* 8) will be as follows:

$$8 + 14 = 22 \rightarrow 22 - 1 = 21 \rightarrow 21 \div 2 = \frac{21}{2} \rightarrow \frac{21}{2} \times 3 = \frac{63}{2} \rightarrow \frac{63}{2} + 2 = \frac{67}{2} \rightarrow \frac{67}{2} \times 6 = 201.$$

In the problem GSS 2.114, the increase, equal to  $-4$ , is then “subtractive.” It is noted as  $4^0$ , as shown below.



Fig 7: GOML-13409 fol. 8, r°. © Government Oriental Manuscripts Library

<i>ga</i>	16	<i>i</i> 7	9	11	12	<i>śe</i> 9	7	5	4
<i>u</i>	$\frac{0}{4}$	$\frac{0}{4}$	$\frac{0}{4}$	$\frac{0}{4}$	$\frac{0}{4}$	$\frac{0}{4}$	$\frac{0}{4}$	$\frac{0}{4}$	$\frac{0}{4}$
<i>ā</i>	64	64	64	64	64	0	0	0	0

In the manuscripts I was able to consult at the GOML, all the quantities like the total sum, the chosen part, the remainder part and the “beginning” of the remainder part, are systematically computed, even if not explicitly asked for in the question. The problem GSS 2.114, for example, is solved by using *sūtra* 2.62 to calculate the total sum, *sūtra* 2.106 for the remainder and the chosen parts, and *sūtra* 2.109 to obtain the values of the “beginnings,” although the question was only about finding the differences.

Only one problem, the last (GSS 2.115), deals with the computation of a difference for a geometrical progression, but in that case, no new algorithm is given: the difference is obtained by subtracting the chosen sum from the total, each of them having been calculated previously.<sup>36</sup>

We have seen that in the first *vyavahāra*, numerous algorithms are provided to answer diverse questions, some of them being “classics,” like finding a sum, a term, a number of terms, etc., and some being less familiar to us today, like those related to the concept of “mixed sums” or “multiple of sums.” However, as I said earlier, the most striking fact is the abundance of computational algorithms given for these two operations.

In the sixth *vyavahāra* (named *miśraka-vyavahāra*), the treatment of series is further expanded. Apart from the sum of squares, sum of cubes and sum of sums usually found in most of the *śreḍhī-vyavahāras*, Mahāvīrācārya deals with arithmetico-geometrical sequences, sums of squares of terms in arithmetical progression and even their cubes, etc.

There is a clear separation between the content of the *parikarman* section and the sixth *vyavahāra*: the *parikarman* represents a sort of “basic” knowledge about series (only arithmetical and geometrical progressions) while the content of the sixth *vyavahāra* is more diverse and advanced. The question is “why did Mahāvīrācārya find it necessary to organize his treatise this way?”

I intend to demonstrate in this paper that the answers may lie in the understanding of the mathematical structure of the Jain cosmos (*loka* or *lokākāśa*). Indeed, after claiming the relevance of the science of computation (*gaṇita*) for many domains of everyday life, Mahāvīrācārya turns in his introductory chapter to the evocation of the Jain universe. He mentions the middle world (*madhya-loka*) with its numerous concentric rings of continents and seas, the higher world (*ūrdhva-loka*) where the gods live, and the lower world (*adho-loka*), abode of the infernal beings, saying that all these worlds are subject to all kinds of quantification and measurement. His conclusion is striking: “What is the good of saying much in vain? Whatever there is in the three worlds, which are possessed of moving and non-moving beings - all that indeed cannot exist as apart from measurement” (GSS 1.16: tr. Rangacharya).

Such a statement is certainly a strong incentive to investigate the mathematical content of cosmological texts, particularly the treatment of series they contain.

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<sup>36</sup> GSS 2.115: *catur-ādi-dvi-guṇātmakottara-yuto gacchaś caturṇām kṛtir daśa-vāñchā padam aṅka-sindhura-giri-dravyendriyāmbhodhayaḥ / kathaya vyutkalitaṃ phalaṃ sakala-sad bhūjāgrimaṃ vyāptavān karaṇa-skandhavan āntaraṃ gaṇitavin mattebha vikṛḍitam // vyutkalita-dhanāni // 258048 // 260096 // 251120 // 261632 // 261888 // 262016 // 26280 //*. M. Rangacharya made a mistake here and gave the chosen sums instead of the remainder sums as the answers, although the manuscripts contain all the correct answers in the text.

## 5. Series in a Cosmological Work, the *Tiloyapaṇṇaṭṭī*

The *Tiloyapaṇṇaṭṭī* (TP) (Exposition of the three Worlds) is an impressive text in the Digambara tradition, written in Prakrit (Jaina Śaurasenī) by Yativṛṣabha (Pkt. Jadivasaha) around the sixth century.<sup>37</sup> With about 5700 *gāthās* and several prose passages, the whole work, divided into nine chapters, supplies ample numerical data relating to the Jaina cosmos and its constituents. Surprisingly, it also contains quite a few general mathematical procedures.

An idea of the global shape of the cosmos and its three main parts is often represented by the image of a cosmic man (*loka-puruṣa*) of colossal dimensions, but a sketch with a tri-dimensional effect can also be helpful:<sup>38</sup>



Fig 8: The world

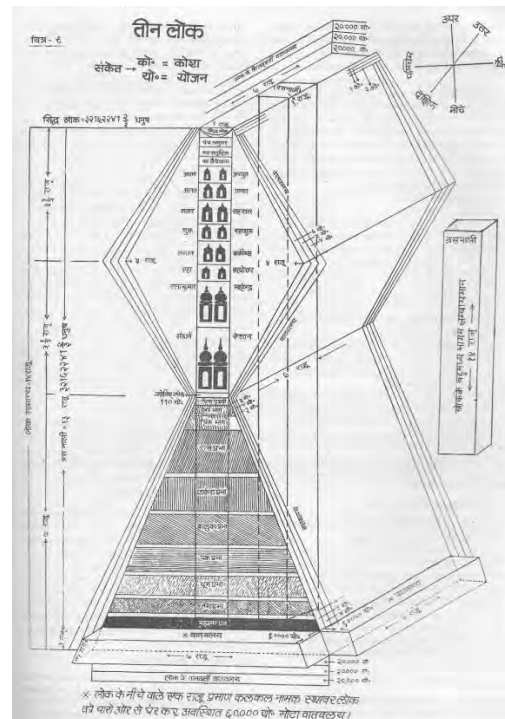


Fig 9: “Tīn Loka”

<sup>37</sup> This date is uncertain but the TP is probably anterior to the GSS. According to Granoff 2009b: 49, “it is agreed that this is the oldest cosmographical text in the Digambara tradition.” In this paper, I will refer to the version in three volumes edited and commented by Āryikā Śrī Viśuddhamatī Mātājī in 1984, 1986, 1988. This edition is based on more manuscripts than the previous one edited by A.N. Upadhye and Hiralal Jain (*Jīvarāja Jain Granthamālā*, Sholapur), and is also available online, on [www.JainGrantha.com](http://www.JainGrantha.com).

<sup>38</sup> Fig. 8: Gouache on paper, XVIth century, Gujarat, in Caillat & Kumar 1981/2004: 53. Fig. 9, in Varṇī 1970 3: 439.

Excellent modern writings with qualitative and quantitative descriptions of different aspects of the Jain cosmos are available.<sup>39</sup> I will concentrate here more on the mathematical aspects.

## The Infernal World

The infernal world is the lower world (*adho-loka*), located below the cosmic man's waist. It is the largest of the three basic regions, with a volume occupying 4/7 of the total cosmos (TP 1.168). It comprises seven realms, shaped like truncated pyramids with a square base that grows larger as one descends, stacked one beneath the other. The Digambaras see the increase in the size of the bases as perfectly regular.

The base is a square whose sides measure one "universe line" (*śreṇī* or *jagaśreṇī*), or 7 "ropes" (*rājus* or *rajjus*). The cosmic man is 14 *rajjus* high, his lower part being 7 *rajjus*. In this lower world, the regular increase in width can, for calculation purposes, be linked to the figure of an isosceles trapezoid or to the concept of arithmetical progression. For example, the widths of the different realms are calculated this way:

TP 1.176: Having subtracted the breadth of the face (Pkt. *muha*) from the breadth of the base (Pkt. *bhūmi*), divided by the height (Pkt. *uccheha*), one has, for each of the earths the [regular] increase for the face or the [regular] decrease for the bottom.<sup>40</sup>

TP 1.117: For any desired [earth], the measure of the decrease or increase is multiplied by its own height (Pkt. *udayā*), this being removed or added [depending on whether one starts] from the bottom or from the face.<sup>41</sup>

Explanation:

Looking at a frontal view of the cosmic man, the width of the face of the lower world is 1 *rajju* and its bottom 7 *rajjus*. The total height is also 7 *rajjus*. Accordingly, the increase (or decrease) is then:  $\frac{7-1}{7} = \frac{6}{7}$  (TP 1.176) and the widths are the successive terms:  $\frac{7}{7}$ ;  $\frac{13}{7}$ ;  $\frac{19}{7}$ ;  $\frac{25}{7}$ ;  $\frac{31}{7}$ ;

<sup>39</sup> See for instance Caillat & Kumar 1981/2004, the essays written by J. Cort, P.S. Jaini, K. Plofker and Ph. Granoff in Granoff 2009a and the sections or chapters on cosmology in Babb 1996: 38-52 and Tatia 1994: 69-90.

<sup>40</sup> TP 1.176: *bhūmīa muhaṃ sohiya uccheha-hidaṃ muhāu bhūmīdo / savvesuṃ khettesuṃ pattekaṃ vaḍḍhi-hāṇō //*

<sup>41</sup> TP 1.177: *ta-kkhaya-vaḍḍhi-pamāṇāṃ ṇiya-ṇiya-udayā-hadaṃ ja-icchāe / hīṇ'abbhahie saṃte vāsāṇi havaṃti bhū-muhāhīṃto //*

$\frac{37}{7}$ ;  $\frac{43}{7}$  and  $\frac{49}{7}$ . The modern formulas  $u_n = 1 + (n-1) \times \frac{6}{7}$  or  $u_n = 7 - (n-1) \times \frac{6}{7}$  with an additive or a subtractive increase, depending on whether one is descending or ascending, would give the same numerical results.

To obtain the volumes, the infernal realms are taken to be equivalent to right prisms with trapezoidal bases. The area of the first one is then  $\left[\left(1 + \frac{13}{7}\right) \div 2\right] \times 1$ , or,  $\frac{10}{7}$  square *rajjus*. The right prism being 7 *rajjus* high, its volume will be 10 (cubic *rajjus*).

The seven volumes are shown in the illustration below where the symbol for the *loka* is  $\equiv$ :

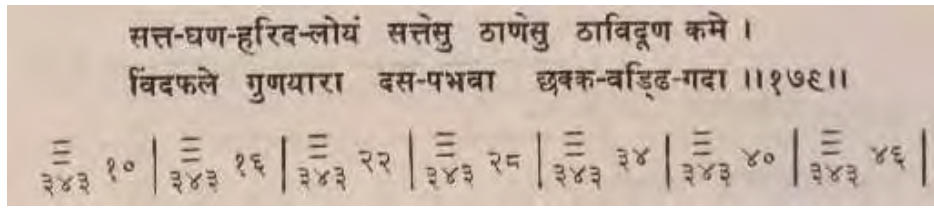


Fig 10: TP vol. 1, p. 49, or p. 126/434 (www.JainGrantha.com)

$\frac{\textit{loka}}{343} \times 10$ ;  $\frac{\textit{loka}}{343} \times 16$ ;  $\frac{\textit{loka}}{343} \times 22$ ;  $\frac{\textit{loka}}{343} \times 28$ ;  $\frac{\textit{loka}}{343} \times 34$ ;  $\frac{\textit{loka}}{343} \times 40$ ;  $\frac{\textit{loka}}{343} \times 46$ , or simply 10, 16, 22, 28, 34, 40 and 46 since the volume of the *loka* is known to be 343 cubic *rajjus* (TP 1.174).

The hellish souls, born in the lower world because of their former crimes or violent acts, suffer innumerable agonies like living in burning heat or intense cold, etc., and many forms of torture and hostility are inflicted upon them. These living conditions become increasingly horrible as one descends towards the lowest hell, which is the gloomiest, darkest and coldest. These realms are named and the list generally starts with the uppermost, the “Gem-hued” (Ratnaprabhā).

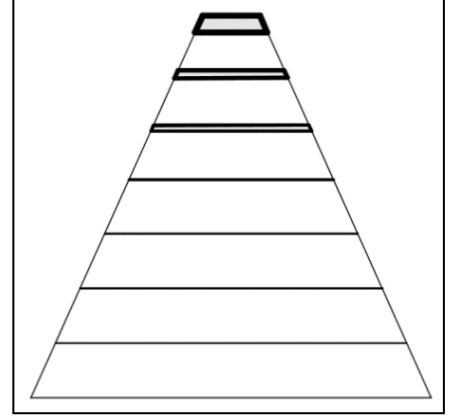
However, the abodes of the hellish beings are located only in horizontal layers in the upper parts of each realm, the total occupied portions having different thicknesses as shown here:

	Namen der Höllen-regionen	Śvetāmbara	Digambara
1	Ratnaprabhā	180 000 Yojana	180 000 Yojana
2	Śarkarāprabhā	132 000 "	32 000 "
3	Vālukāprabhā	128 000 "	28 000 "
4	Pañkaprabhā	120 000 "	24 000 "
5	Dhūmaprabhā	118 000 "	20 000 "
6	Tamahprabhā	116 000 "	16 000 "
7	Tamastamahprabhā	108 000 "	8 000 "

Table 3: Kirfel 1920: 316

Now, a thickness of 180 000 *yojanas* is almost negligible against a height of 1 *rajju*, which is an unfathomable unit of length (an *asaṃkhyāta* quantity), larger than any expressible number.<sup>42</sup> The depiction given in a familiar sketch, like the three-dimensional one above (Fig. 8), is then misleading as the portions exempt of hellish souls seem to be the extremely tiny white horizontal strips separating the seven regions, while the coloured parts seem to occupy all the space. What should have been depicted is exactly the opposite; something more like this image, where the occupied parts are extremely tiny and the whole image looks empty (Fig. 11):

However, to show here mostly empty spaces wouldn't have been very useful. We should not forget that the illustrations were made for teaching purposes and, wanting to draw the viewers' attention to specific points of interest, artists would not have hesitated to ignore the proportions, if necessary.



The upper and middle parts (*khara-bhāga* and *pañka-bhāga*) of Ratnaprabhā are the abode of Vyantara and Bhavanavāsin deities, but the lowest part, as well as the six earths below, are the abode of hellish souls. These are born and live in huge “holes” or “cavities” (Pkt. *ñiraya-bila*), respectively 3 000 000, 2 500 000, 1 500 000, 1 000 000, 300 000, 99 995 and 5 of them, or a total of 8 400 000 or 84 lakhs for the seven hells, all located in a kind of vertical “tunnel” (*trasanāli*) traversing the entire cosmos.<sup>43</sup>

Each hell is divided into a certain number of layers that decreases from top to bottom: 13, 11, 9, 7, 5, 3, 1. The total is 49. Each layer has a central or main hole (*indraka*) and a set of aligned (*śreṇībaddha*) holes located on the eight lines radiating from the centre towards the four main directions and the four intermediary minor ones. There are also numerous “scattered” (*prakīṛṇaka*) or non-aligned holes.

The first *indraka* is named Sīmanta, and there are as many *indrakas* as there are layers, hence a total of 49. These layers have to be shown in the illustrations since their existence gives rise to an intense mathematical activity, hence the most familiar image of the hellish world is the one shown below where the layers can be seen and even counted:<sup>44</sup>

<sup>42</sup> Jain theoreticians have devised a threefold classification of numbers into numerable (*saṃkhyāta*), innumerable (*asaṃkhyāta*) and infinite (*ananta*). See for instance Singh 1988 for more details.

<sup>43</sup> The *trasanāli* can be visualized in Fig. 8 & 9.

<sup>44</sup> In Varṇī 1970 3: 441.



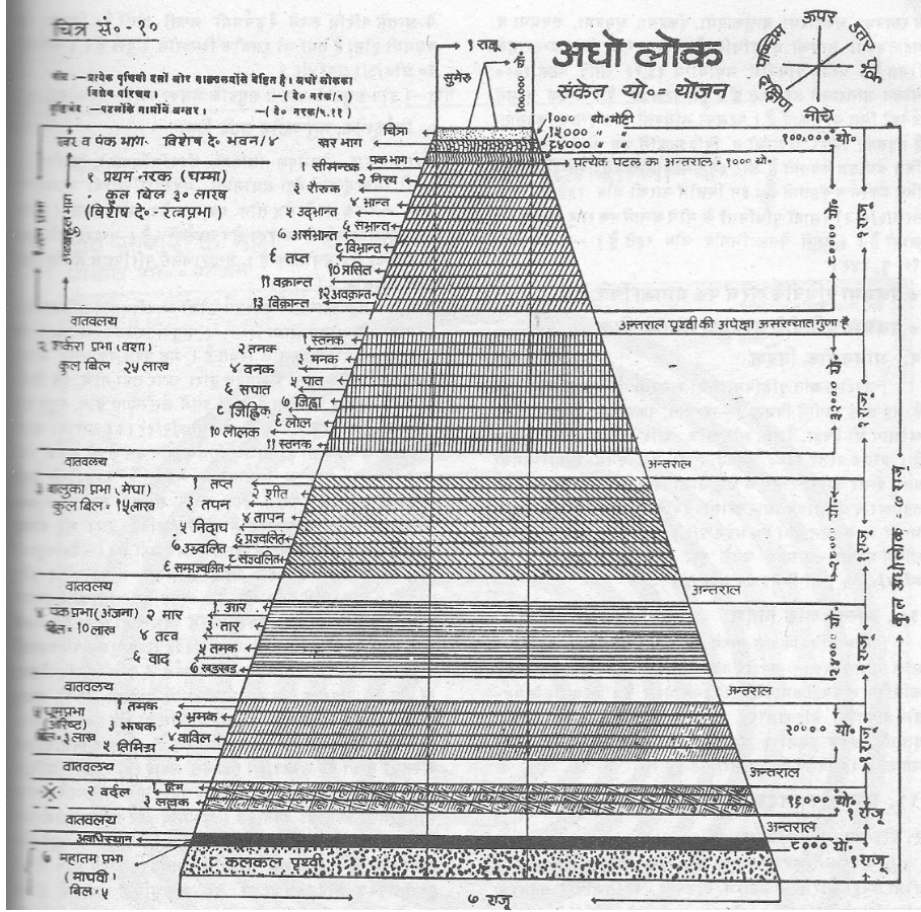


Fig 12: "Adho Lok"

In the first layer of Ratnaprabhā there are 49 aligned (*śreṇībaddha*) holes in each of the four main directions, and 48 in the intermediate ones, hence a total of  $49 \times 4 + 48 \times 4 + 1$  or 389 aligned holes, including the central one in the first layer (TP 2.55). The numbers in each direction decrease by 1 for every layer situated below, so the total number diminishes by 8 every time and only 5 holes remain in the last layer (TP 2.56-57).

The determination of the total number of *śreṇībaddha* holes in any layer is provided by two algorithms, depending on whether one starts from Ratnaprabhā or from Tamastamaḥprabhā at the bottom. These are identical to the first part of GSS 2.64:

TP 2.58: The rank of a chosen *indraka* is diminished by *one*, multiplied by 8 and, according to the rule this is subtracted from 389; the rest is the number of holes in this layer [including the central one].<sup>45</sup>

<sup>45</sup> TP 2.58: *iṭṭh'-iṃḍaya-ppamāṇaṃ rūūṇaṃ aṭṭha-tāḍiyā ṇiyamā / uṇa-ṇavadīti-saesu avaṇiya seso havaṃti ta-ppaḍalā //*

TP 2.59: The rank chosen is removed from 49; [the rest], according to the rule, is multiplied by 8 and 5 is added. This is the number of *śreṇībaddha* holes including the *indraka*.<sup>46</sup>

Explanation:

Let  $k$  be the rank of a chosen *indraka* among the 49 and  $(u_n)$  the sequence giving the numbers of aligned holes, with the central one for a rank  $n$ .

If  $u_1 = 389$  then  $u_k = u_1 + (k - 1)r$ , with  $r = -8$ , this corresponds to the algorithm TP 2-58:

$$k \rightarrow k - 1 \rightarrow (k - 1) \times 8 \rightarrow 389 - (k - 1) \times 8 \rightarrow u_k$$

If the last one becomes the “beginning,” meaning if  $u_1$  becomes  $u_{49}$  which is equal to 5, then  $u_k = u_{49} + (49 - k)r$  and  $r = 8$ , hence the algorithm TP 2.59:

$$k \rightarrow 49 - k \rightarrow (49 - k) \times 8 \rightarrow (49 - k) \times 8 + 5 \rightarrow u_k$$

The inverse question is also of interest: knowing the total number of aligned and central holes in one layer, what is the rank of the central hole? The answer is provided in TP 2.60.<sup>47</sup>

Other questions also appear, for example, finding the total number of these kinds of holes for any chosen earth. There are at least two ways to proceed in this case, the first is to calculate the “beginning” of the chosen earth (a value among these:  $u_{13}$ ,  $u_{13+11}$ ;  $u_{13+11+9}$ ; ...;  $u_{13+11+\dots+1}$ ), with the help of TP 2.58, and then to apply the algorithm in TP 2.64:

TP 2.64: The number of terms (*pada*) lessened by the rank of the chosen earth is multiplied by the increase; the product of the chosen rank lessened by *one*, and the increase, is added; the double of the “beginning” is added; [the result] is multiplied by half the number of terms; we have the total sum.<sup>48</sup>

In modern notation, for any “desired” hell (rank  $d$ ) which has  $\alpha$  layers and a “beginning” which is known, the total sum  $S$  is obtained by calculating these successive steps, with  $r = 8$ :

$$(\alpha - d) \times r$$

$$(\alpha - d) \times r + (d - 1) \times r$$

<sup>46</sup> TP 2.59: *icche padara-vihīṇā uṇavaṇṇā aṭṭha-tāḍiyā ṇīyamā /sā paṃca-rūva-juttā icchida-seḍh'-iṃdayā hoṃti //*

<sup>47</sup> TP 2.60: *uddiṭṭhaṃ paṃc'-ūṇaṃ bhajidaṃ aṭṭhehi sodhae laddhaṃ / eguṇavaṇṇāhiṃto sesā tath' iṃdayā hoṃti //*

<sup>48</sup> TP 2.64: *caya-hadam icch 'ūṇa-padaṃ rūvūṇ 'icchāe guṇida-caya-juttaṃ / du-guṇida-vadaṇeṇa judaṃ pada-dala-guṇidaṃ havedi saṃkalidaṃ //*



$$(\alpha - d) \times r + (d - 1) \times r + 2 \times \text{beginning}$$

$$[(\alpha - d) \times r + (d - 1) \times r + 2 \times \text{beginning}] \times (n / 2) \rightarrow S$$

The “beginnings” are successively 293, 205, 133, 77, 37, 13 and 5 (TP 2.62). After reduction, the formula obtained in TP 2.64 is the same as that provided in GSS 2.62:

$$[(\alpha \times r - r + 2 \times \text{beginning}] \times (\alpha / 2) \rightarrow S).$$

It shows that the introduction of  $d$  was not a mathematical necessity. However, the values of  $\alpha$  and of  $d$  being linked, it was presumably useful to state them both, before applying the algorithm: this is what the prose passage following TP 2.64 seems to suggest:<sup>49</sup>

The number of terms (*pada*) less the [rank of the earth] chosen is multiplied by the increase (*caya*)  $\frac{1}{13} | 8 | [ \dots ]$ .

Writing  $\frac{1}{13}$  seems to work as a reminder, giving at the same time the rank of the earth and the number of its layers.<sup>50</sup>

### An Interesting Algorithm

Another way to calculate the total number of aligned and central holes of a chosen earth is provided in TP 2.6, in an interesting algorithm since, apart from knowing that the increase is 8 and the number of such holes in the seventh earth is 5, only the knowledge of the number  $\alpha$  of layers is required to arrive at the answer: there is no need to know the “beginning.”

TP 2.65: The number of central holes of a chosen earth lessened by *one* is divided by two and squared; one adds its root, multiplies by 8 and adds 5; then, one multiplies by the number of central holes [of the earth]; one has the total sum for the earth.<sup>51</sup>

In modern notation:

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<sup>49</sup> *caya-hadam icch 'ūṇa-padaṃ*  $\frac{1}{13} | 8 |$ .

<sup>50</sup> This should not be understood as a fraction as it just means that the calculation to be done here is the difference 13 - 1.

<sup>51</sup> TP 2.65: *ekkoṇamavaṇi-iṃdayam addhiya vaggejja mūla-saṃjuttaṃ / aṭṭha-guṇaṃ pañca-judaṃ puḍhaviṃdaya-tādidammi puḍhavi-dhaṇaṃ ||*

$$\begin{aligned} \left(\frac{\alpha-1}{2}\right)^2 &\rightarrow \left(\frac{\alpha-1}{2}\right)^2 + \frac{\alpha-1}{2} \rightarrow \left[\left(\frac{\alpha-1}{2}\right)^2 + \frac{\alpha-1}{2}\right] \times 8 \rightarrow \left[\left(\frac{\alpha-1}{2}\right)^2 + \frac{\alpha-1}{2}\right] \times 8 + 5 \rightarrow \\ &\rightarrow \left[\left[\left(\frac{\alpha-1}{2}\right)^2 + \frac{\alpha-1}{2}\right] \times 8 + 5\right] \times \alpha. \end{aligned}$$

The results, stated in TP 2.66-68, are 4433, 2695, 1485, 707, 265, 63 and 5. This algorithm is not only interesting in itself, it also provides a beautiful opportunity to use the operation of subtraction (*vyutkalita*) to justify it.<sup>52</sup> Indeed, the total number of this kind of holes in a chosen earth can be obtained by the difference between the total number of holes up to this earth, and the total number, up to the preceding one.

Explanation:

Let us call  $\alpha$  the number of layers in a chosen earth. The preceding earth will have  $\alpha - 2$  layers. Using the same modern notation as for the *vyutkalita* section in this paper, one has to compute the equivalent of  $n + p - 1$ , to start the algorithm.

$$\begin{aligned} \text{Here, } n + p - 1 &= [(1 + 3 + 5 + \dots + (\alpha - 2)) + \alpha] + [(1 + 3 + 5 + \dots + (\alpha - 2))] - 1 \\ &= 2[(1 + 3 + 5 + \dots + (\alpha - 2))] + \alpha - 1. \end{aligned}$$

$$\text{And, accordingly, we get: } \frac{n+p-1}{2} = [(1 + 3 + 5 + \dots + \alpha - 2)] + \frac{\alpha - 1}{2}.$$

The means of obtaining the sum of the  $n$  first odd integers has long been known in India.<sup>53</sup> In modern notation, we have the formula:  $1 + 3 + 5 + \dots + (\alpha - 2) = \left(\frac{\alpha - 1}{2}\right)^2$ . Hence, the final value

of  $\frac{n+p-1}{2}$  is equal to  $\left(\frac{\alpha - 1}{2}\right)^2 + \frac{\alpha - 1}{2}$  and the first part of the algorithm TP 2.65 is justified, the rest being easy to prove.

TP 2.70 gives a general procedure for the calculation of the total number of central and aligned holes in the entire hellish world: there are 9653 central and aligned holes (TP 2.73) and 9604 aligned ones (TP 2.82).

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<sup>52</sup> Today, we could also justify it by using the mathematical principle of induction, as shown in Jadhav 2005; however, ancient mathematicians did not explicitly state the inductive hypothesis and their reasoning was not having the necessary rigor.

<sup>53</sup> See Datta & Singh 1993: 103-5.

One of the questions that appear next is about finding the number of layers in an earth, when the number of aligned holes is known. The *gāthās* TP 2.85-86 provide a general answer, very much like GSS 2.70, which gives the procedure to calculate the number of terms in an arithmetical progression. Indeed, after simplification the formulas are similar:<sup>54</sup>

For example, we can compare the two formulas:  $\left(\sqrt{8Sr + (2u_1 - r)^2} - (2u_1 - r)\right) \div (2r) = n$  (in GSS 2.70), and  $\left(\sqrt{S \times r/2 + \left(\frac{u_1 - r/2}{2}\right)^2} - \left(\frac{u_1 - r/2}{2}\right)\right) \div (r/2) = n$  (in TP 2.85).

After dealing with the central and aligned holes, procedures are given to carry out similar kinds of computations only regarding aligned holes and, by subtraction from the total number of holes, the number of scattered (*prakīrṇaka*) holes is obtained (TP 2.88-94).

earth	total number of holes	central	aligned	scattered
1 <sup>st</sup>	3 000 000	13	4420	2 995 567
2 <sup>nd</sup>	2 500 000	11	2684	2 497 305
3 <sup>rd</sup>	1 500 000	9	1476	1 498 515
4 <sup>th</sup>	1 000 000	7	700	999 293
5 <sup>th</sup>	300 000	5	260	299 735
6 <sup>th</sup>	99 995	3	60	99 932
7 <sup>th</sup>	5	1	4	0

Table 4: Number of holes in the lower world

Numerous numerical data concerning the shapes and dimensions of these three categories of holes are provided next: the horizontal and vertical intervals between them, the number of hellish beings living there, their minimum and maximum ages, their heights, etc., this information is spread over nearly two hundred stanzas (TP 95-271). Some operations require the use of arithmetical progressions, while others do not.

For example, it is stated that one fifth of the holes have a numerable (*saṃkhyeya*) extent and the rest, four-fifths of the total, have an innumerable one (*asaṃkhyeya*) (TP 2.96). There is a *saṃkhyeya* quantity of infernal beings in holes having a *saṃkhyeya* extent and an *asaṃkhyeya* quantity in those of *asaṃkhyeya* extent (TP 2.104).

It can also be observed that the widths (Pkt. *rumda*, *vitthāra*) of the *indrakas* are in arithmetical progression or regression, knowing that Sīmanta, the first *indraka* in Ratnaprabhā,

<sup>54</sup> TP 2.85: *caya-dala-hada-saṃkalidaṃ caya-dala-rahidādi addha-kadi-juttaṃ / mūlaṃ parimūl'-ūṇaṃ pacayaddha-hidammi taṃ tu padam //*

measures 45 lakh (*i.e.* 4 500 000) *yojanas*, which is the diameter of the human world (*manuṣya-loka*), and Avadhīsthāna, the last one in the lowest hell, measures 1 lakh *yojanas*, which is the diameter of Jambūdvīpa.<sup>55</sup>

The increase or decrease in any *indraka* and its width can then be computed with the help of TP 2.105-207 since  $u_{49} = 4500000$ ,  $u_1 = 100000$  and  $u_n = u_1 + (n-1) \times r$  if one works upwards. We get:  $r = \frac{4500000 - 100000}{49 - 1} = \frac{4400000}{48} = 91666 + \frac{2}{3}$ .

TP 2.105: The first [central hole] has [an extent] of forty-five lakhs [*yojanas*], the last central hole (*indraka*) 1 lakh; [this] is removed from the other, the increase-decrease is [obtained] by the division by the [total of] central holes lessened by one.<sup>56</sup>

TP 2.106: The measure of the increase-decrease is 91 thousand and 666 *yojanas*, and the fraction 2 divided by 3 (*du-kalāo ti-vihattā*).<sup>57</sup>

TP 2.107: [To find the extent of an *indraka* starting] from the second one, the increase-decrease is multiplied by the chosen rank lessened by one; this is removed from [the extent of] *Sīmanta* or added to the one of Avadhīsthāna.<sup>58</sup>

The following stanzas (TP 2.108-156) provide the extents of the 49 *indraka* holes, one after the other, as well as their heights, their horizontal and vertical intervals, etc., these results are obtained using different patterns. The minimum and maximum ages of hellish beings in the different layers of each earth is also discussed. For example, the maximum age of hellish beings in the first two layers of Ratnaprabhā is said to be a numerable number of years, but in all the layers below it, it is an innumerable, constantly increasing quantity.<sup>59</sup>

While investigating the content of the GSS above, I had mentioned that no author, apart from Mahāvīrācārya, ever remarked on the possibility of obtaining the same value for the total sum in the case of an arithmetical progression, when the order of the terms is reversed. It seems

<sup>55</sup> The human world is made up of two and a half continents, hence its usual name, Aḍhādvīpa. The choice of values here is certainly not a mere coincidence.

<sup>56</sup> TP 2.105: *paṇadālaṃ lakkhāṇiṃ paḍhamo carim iṃdao vi igi-lakkhaṃ / ubhayaṃ sohiya ekkoṇ' iṃdaya bhajidammi hāṇi-cayaṃ //*

<sup>57</sup> TP 2.106: *dāvattḥi-chassayāṇi igiṇaudi-sahassa joyaṇṇāṇi pi / du-kalāo ti-vihattā parimāṇaṃ hāṇi-vaḍḍī //*

<sup>58</sup> TP 2.107: *vidiyādisu icchaṃto rūṇicchāe guṇida-khaya-vaḍḍī / sīmaṇtādo sohiya melijja suavahi-ṭhāṇammi //*

<sup>59</sup> In the last layer of Ratnaprabhā it is 1 *sāgaropama*, and in the very last hell, 33 *sāgaropamas* (TP 2.204).

that the situation prevalent in the seven hells, where the flexibility of moving upwards or downwards is often skillfully used, may have been a source of inspiration for Mahāvīrācārya.

## The Middle World

The middle world (*madhya-loka*) is located right above the infernal world, like a horizontal lid. Shaped like an upright cylinder of huge diameter (1 *rajju*), but of negligible height, it comprises a succession of alternating rings of land and rings of sea surrounding each other. However, it is not always visible in illustrations, like Fig. 9, where it is only represented by a line. In some paintings, when the artist wants to show some details of the innermost and central land called Jambūdvīpa, he presents the image frontally, as in Fig. 8, after having rotated the axis of the cylinder.<sup>60</sup>

The fourth chapter of the TP is devoted to the depiction of the first “two-and-a-half” continents (Aḍhādvīpa) of the middle world, over nearly 3000 *gāthās*, and it is by far the most extensive chapter of the work. The importance attributed to it is probably because this is the only part of the cosmos humans can inhabit.<sup>61</sup>

The fifth chapter deals with the set of alternating rings of continents or islands (*dvīpa*) and oceans (*samudra*) surrounding the Jambūdvīpa. The last ring is an ocean called Svayaṃbhūramaṇa. Its outer border delineates the limit between the bounded cosmos (*loka*) in the middle world and the infinite space or trans-cosmos (*a-loka*) that surrounds it and which is totally exempt of substances (*dravya*).

The width of the succeeding concentric rings (*valaya*) doubles at every step, with no gaps. It means that, just as in the case of the nether world, the illustrations cannot possibly be to scale, as a Jambūdvīpa measuring a few millimeters would barely leave room for the representation of the Aḍhādvīpa on an A4 sheet.

As I mentioned earlier, illustrators constantly had to bear in mind the purpose of their illustrations and had to make choices. Here, for the representation of the middle world, they had to choose between either to create a sensation of the innumerability of the rings, and forget about the fact that the width doubled constantly, as shown on the left side of the painting below, or to depict some details of the geography of the Jambūdvīpa. In the latter case they had

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<sup>60</sup> Jambūdvīpa is often shown with details, as its southern part is Bhārata (India). The whole island is divided into seven regions, separated by six parallel mountain ranges running from east to west. The distance between them doubles up to the centre and then it is divided by two, from south to north. For more details, see for example Van Den Bossche 2007.

<sup>61</sup> A description of the Aḍhādvīpa is given in Balbir 2008 and information related to the rings of oceans and islands as well as the sacred mountains is provided for instance in Hegewald 2000.

to vastly minimize the size of the Salted Ocean (Lavaṇasamudra) surrounding the Jambūdvīpa, as shown on the right side.<sup>62</sup>

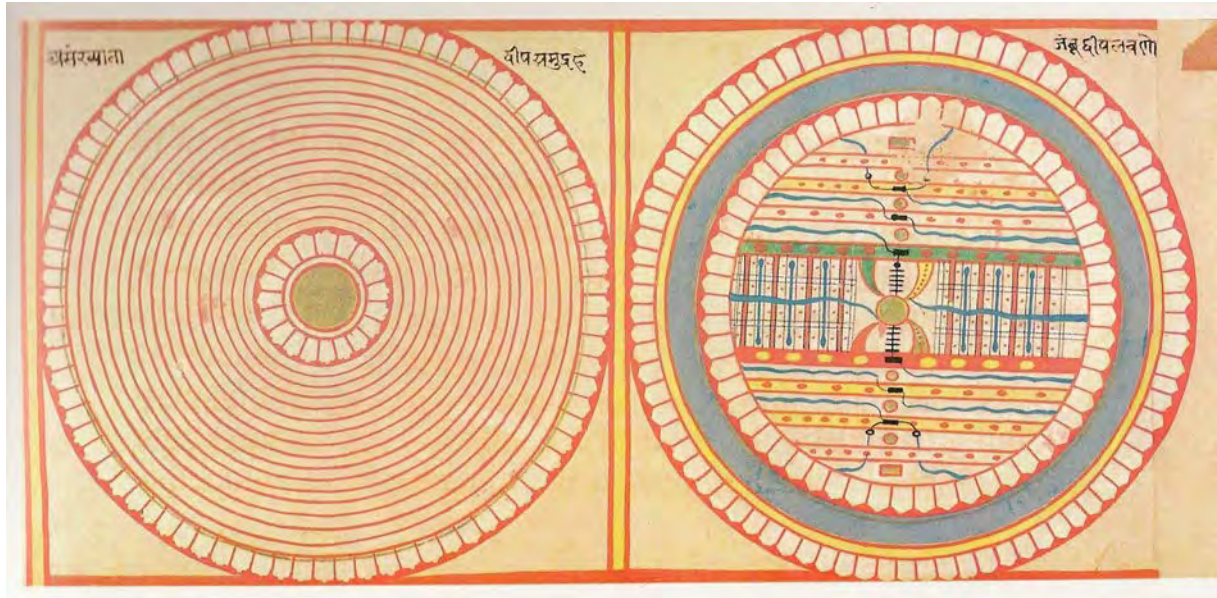


Fig 13: The islands (*dvīpas*) and oceans (*samudras*) of the middle country

### Widths, Diameters and Areas

The main mathematical tool used in this chapter relies on algorithms linked to geometrical progressions, since the width doubles each time.

With the same notations as those used in Gupta (1992a-b),  $(W_n)$  being the sequence that provides the width (Pkt. *vāsalvikkhaṃbhalvitthāra/rūṇḍa*) of the rings, starting from  $W_0 = D = 100\,000$  *yojanas* or 1 lakh *yojanas* for the Jambūdvīpa, one can easily arrive at the explicit formula  $W_n = 2^n D$ .

However, a major difficulty is present throughout this chapter as the number of rings is an innumerable (*asaṃkhyāta*) quantity: this prevents a direct application of the usual formulas related to geometrical progressions, to arrive at numerical answers for the final rings. Hence other methods will have to be found. This is precisely the case here, where new results flourish throughout the chapter, giving it an amazing mathematical flavour.

Two of the main stanzas used to by-pass the difficulty mentioned above, are as follows:<sup>63</sup>

<sup>62</sup> Gouache on paper, XVIII<sup>th</sup> century, Rajasthan, in Caillat & Kumar 1981/2004: 107.

<sup>63</sup> TP 5.33: *bāhira-sū-majjhe lakkha-tayaṃ melidūṇa caū-bhajidel icchiya-dīvaḍḍhīṇaṃ vitthāro hodi valayāṇaṃ* // TP 5.34: *lavaṇādīṇaṃ ruṇḍaṃ du-ti-cau-guṇidaṃ kamā ti-lakkhūṇaṃ / ādima-majjhima-bāhira-sūṇaṃ hodi parimāṇaṃ* // From there, many more results can be obtained. See Gupta 1992a.

TP 5.33: Adding three lakhs (Pkt. *lakkha*) to the external diameter of a chosen continent or ocean, dividing by four, one gets the width of the [chosen] ring.

TP 5.34: Starting with the ocean Lavaṇa, the width [of a continent or an ocean] is multiplied in this order by 2, 3 or 4; by removing three lakhs from the result, one gets the interior, median and exterior diameter [of that ring].

Explanation:

If one calls  $(I_n)$ ,  $(C_n)$  and  $(D_n)$  the sequences giving the inner diameter (Pkt. *ādima-sūī*, Skt. *ādya-sūcī*) of a ring, its median diameter (Pkt. *majjhima-sūī*, Skt. *madhyam-sūcī*) and its outer diameter (Pkt. *bāhira-sūī*, Skt. *bāhya-sūcī*), these results can be established:

$$\begin{aligned} D_n &= W_0 + 2(W_1 + W_2 + \dots + W_n) = D + 2D(2^1 + 2^2 + \dots + 2^n) \\ &= D + 2D \times 2 \times \frac{2^n - 1}{2 - 1} = D + 4D(2^n - 1) = D + 4W_n - 4D = 4W_n - 3D. \end{aligned}$$

From  $D_n = 4W_n - 3D$ , one gets  $W_n = (D_n + 3D) \div 4$ , the result stated in TP 5.33. Then, knowing that  $I_n = D_{n-1}$  and  $C_n = (I_n + D_n)/2$ , one can also write  $I_n = 2W_n - 3D$  and  $C_n = 3W_n - 3D$ . R. C. Gupta pointed out that TP 5.34 gives three algorithms in a single stanza and does it “in a remarkably concise way”:<sup>64</sup>

$$I_n = 2W_n - 3D, C_n = 3W_n - 3D, D_n = 4W_n - 3D.$$

Whenever the rank  $n$  of a ring is known, all these parameters can be computed, as shown in Table 5 below, regarding the widths and diameters of the first rings (in *yojanas*):

$n$	Name	$W_n$	$I_n$	$C_n$	$D_n$
0	Jambūdvīpa	1 lakh			1 lakh
1	Lavaṇasamudra.	2 lakhs	1 lakh	3 lakhs	5 lakhs
2	Dhātākīkhaṇḍadvīpa	4 lakhs	5 lakhs	9 lakhs	13 lakhs
3	Kālosamudra	8 lakhs	13 lakhs	21 lakhs	29 lakhs
4	Puṣkaradvīpa	16 lakhs	29 lakhs	45 lakhs	61 lakhs
5	Puṣkaravaravāridhi	32 lakhs.	61 lakhs	93 lakhs	125 lakhs

<sup>64</sup> See Gupta 1992a: 91. In modern notation, we would also write:  $I_n = (2^{n+1} - 3)D$ ,  $C_n = (2^n - 1) \times 3D$  and  $D_n = (2^{n+2} - 3)D$ .

The last numerical value of  $n$  being unknown, the formula  $W_n = (D_n + 3D) \div 4$  will be used to find the width of the last ocean, the Svayaṃbhūramaṇasamudra, as its outer diameter  $D_n$  is known: 1 *rajju*.

The result is stated as:  $\frac{1}{4}$  *rajju* +  $\frac{3}{4}$  lakh of *yojanas* or  $\frac{1}{28}$  *jagaśreṇi* + 75000 *yojanas* since one *jagaśreṇi* is equal to 7 *rajjus*.

From there, it is easy to obtain its inner and mean diameters. Then, dividing the width regularly by two, the widths of the last continent (Svayaṃbhūramaṇadvīpa), the penultimate ocean (Ahīndravarasamudra) and the penultimate continent (Ahīndravardvīpa) etc., can be obtained, working backwards. Their diameters can also be assessed.

Another important subject of discussion in this chapter is the calculation of the areas ( $A_n$ ) of the islands and seas. In the GSS, Mahāvīrācārya puts forward some rules (GSS 7.28 and 7.67 ½) to obtain the “partially approximate” (*bādara*) and the “minutely accurate” (*sūkṣma*) values of areas of an inlying (*abhyantara*) or an out-reaching (*bahis*) annular figure (*cakravāla-vṛtta*), when the outer or the inner diameter is known.<sup>65</sup>

In modern notation, if  $W$  is the width of a ring,  $I$  its inner diameter and  $D$  the outer diameter, the area is  $\pi \times (I + W) \times W$  or  $\pi \times (D - W) \times W$ . Now, as a very special case and with the help of the formula  $I_n = 2W_n - 3D$ , the area of any ring can be obtained solely from the knowledge of its width:  $A_n = 3\pi(W_n - D) \times W_n$ .

Then, as stated in TP 5.244, the “gross” value (taking  $\pi = 3$ ) of the area of the  $n^{\text{th}}$  ring is:  $A_n = 9(W_n - D) \times W_n$ . Again, from this result, the area of the last ocean Svayaṃbhūramaṇa can be obtained directly since its width is known, as can the areas of the continents and oceans preceding it, if required.<sup>66</sup>

Other features related to the geometry of the annular figure are inspected in the same chapter, for example  $K_n$  is the number of “sections/divisions” (Pkt. *khaḍa*, Skt. *khaṇḍa*), i.e. the number of times the area of the Jambūdvīpa is comprised in a ring’s area. The formula

$K_n = \frac{D_n^2 - I_n^2}{D^2}$  can be obtained from TP 5.36 and it gives:

$$K_0 = 1, K_1 = 24, K_2 = 144, K_3 = 672, \text{ etc. }^{67}$$

<sup>65</sup> These are the technical terms Rangacharya uses in his translation. The “partially approximate” and “minutely accurate” values are obtained by taking 3 or  $\sqrt{10}$  for the value of  $\pi$ . Mahāvīrācārya is the only author, with Nārāyaṇa Paṇḍit (c. 13<sup>th</sup> century) to have studied this geometrical figure (Sarasvati 1979: 170)

<sup>66</sup> Calculations concerning the last ocean are done because it is the last one, of course, but probably also because it is the only place in the universe apart from the first two oceans where aquatic life is present (TP 5.31).

<sup>67</sup> TP 5.36: *bāhira-sūi-vaggo abbhaṃtara-sūi-vagga-parihīṇo / lakkhassa kadimmi hide icchaya-dīvuvahi-khaṇḍa-parimāṇam* // See Gupta 1992a for more results.



There are many more interesting and non-trivial cases of computation carried out in this fifth chapter, which require a good mastery over the handling of geometrical progressions, but it is not possible to do justice to them in this paper. The most striking example, in this respect, is the systematic approach used to deal with “the nineteen cases (*vikalpa*) of comparison” for which, in each case, algorithms are given to obtain the answers.<sup>68</sup>

Other regions of the cosmos also provide numerous opportunities for the application of arithmetical and/or geometrical progressions. For example, when evaluating the number of moons and other luminous deities evolving in the region just above the rings of the middle world and below the upper world, both are required.<sup>69</sup>

For the description of the heavenly or upper world (*ūrdhva-loka*), mostly arithmetical progressions are at play, but to a much lesser extent than in the hellish region.

## 6. Conclusion

Series were by no means the only mathematical tool Jain thinkers used to describe the cosmos and quantify its constituent parts.<sup>70</sup> However, even if only short extracts of the TP have been presented here, I think it is possible to appreciate the extent and importance this tool seems to have had.

In these circumstances, and to answer the question raised by K. Plofker, cited in the introduction to this paper, I would say that Mahāvīrācārya’s “quite daring” act of casting out the basic addition and subtraction of two numbers, and of replacing them with more sophisticated operations relating to series, seems to have resulted from a deliberate and relevant choice made in agreement with the Jain context. Indeed, these two canonical operations probably seemed too simple and of very limited utility in comparison to what is done with series in the description of the Jain cosmos. Furthermore, the new operations Mahāvīrācārya created and named *saṃkalita* and *vyutkalita* must have seemed to him to necessarily belong to the “basic” or “pre-requisite” knowledge students had to master. They therefore had to be part of the first “practice” or “procedure” (*vyavahāra*) of the GSS, the one dealing with the

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<sup>68</sup> For example, in the 5th case, for any chosen continent, the difference between its width and the width of the preceding continent is asked for.

<sup>69</sup> Luminous deities are supposed to evolve or be stationary in an imaginary plane surface parallel to the lands and seas of the middle world. The number of moons doubles on every first sub-division of each successive ring and then increases by four on each successive sub-division inside that very ring. See Jain 1995 for more details.

<sup>70</sup> Much could be said also on the geometry of the circle and chord segments, on operations like exponentiation and roots of various degrees, on the astounding recursive processes leading to the definitions of the different innumerable (*asaṃkhyāta*) and infinite (*ananta*) quantities, etc. Examples are many more. See for example Singh 1988 and Jain 2008-2009.

“operations” or “preparations” (*parikarman*), instead of being included only in an usual and specific “practice on series” (*śreḍhī-vyavahāra*).

Probably neither D.E. Smith, nor Professor Rangacharya could have imagined that such a question would be raised one day, as Jain Studies were in an embryonic state at the beginning of the 20<sup>th</sup> century and no detailed translations of Jain texts on cosmology had as yet been edited or published.<sup>71</sup> Also, historians of mathematics at that time were focused mainly on questions concerning the supposed influence of Greek mathematics on “Hindu” mathematics, the question of the origin of Indian algebra being particularly crucial and at the heart of intense debates dividing “non-believers” and “believers.”<sup>72</sup>

As we can judge from the introduction to the GSS he wrote, D.E. Smith certainly was a “believer,” and he even considered the GSS a key text in that respect.<sup>73</sup> However, he had no clue that the Jain understanding of the universe could also have had such a strong impact on the structure of the text itself.

There is no explicit mention of this idea in any Jain text, but we may reasonably suppose that the choice of putting a strong emphasis on arithmetical and geometrical progressions, among all the possible types of progressions, to describe many elements of the cosmos is not without reason. They both convey a high degree of stability and predictability since, by definition, increases are always regular; in addition, their mathematical patterns are not difficult to grasp, even by non-mathematicians, and they surely have a soothing and reassuring effect.<sup>74</sup>

It is known that arithmetical progressions don’t increase as fast as geometrical ones, when the ratio is larger than *one*, but we can observe that it is not seen as a hindrance to the description of the hellish regions, as it is the unfathomably large gaps between the different realms that create the astounding vastness of this part of the cosmos. Even in the case of the middle region, the fact that the width of the rings increases very rapidly, and that the number of

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<sup>71</sup> The initial difficulties met by scholars to access to the Jain *bhaṇḍāras* and to get publications done, as well as the recent increasing enthusiasm for Jaina Studies amongst the wider public and amongst professional academics are evoked in Flügel 2005: 2-8.

<sup>72</sup> See Heeffer 2009 for more details. The historian Moritz Cantor was a major ‘non-believer’. He spread the idea that “Indian knowledge must have stemmed from the Greeks” (Heeffer 2009: 5).

<sup>73</sup> According to D. E. Smith: “there is no evidence of any considerable influence of Greek algebra upon that of India. The two subjects were radically different.” He also adds that Professor Rangacharya confirmed that “India developed an algebra of her own, [...], India influenced Europe in the matter of algebra, more than it was influenced in return [...].” (Smith 1912: xxi). It should be added here that the “algebra” found in the GSS is not a symbolic one but is rather a collection of proto-algebraic recipes to solve many kinds of problems, like the ones in chapters 4 and 6.

<sup>74</sup> Granoff 2009b: 49-63 rightly contrasts the chaos and the unpredictability governing the world of transmigration (*saṃsāra*) with the order and safety imposed by the structure of the cosmos.

rings is innumerable, is overcome by the fact that the outer diameter of the last ring is known. All the useful calculations can then still be carried out, for example those concerning the width and the area of the last ocean.

By choosing such a model for the cosmos, Jain thinkers succeeded in combining seemingly contradictory ideas like, on the one hand, a vastness which is beyond our imagination and, on the other hand, a regularity and perfect command over the calculations, which is maintained in all circumstances. But, at the same time and most importantly, there is a point that should be kept in mind, as J. Cort (2009: 42) rightly remarks:

There is a point to this vastness, and Jain teachers want to drive home this point to the faithful: the universe is physically vast, and the portion of it where humans can follow a fully religious life is almost negligible. [...] The message of the mind-numbing lesson of Jain cosmology is that a person should understand just how precious and rare this human life is and make the best of it.

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