Risk Management: Principles and Applications

Unit 1 Introduction to Risk Management

Contents

Unit Overview

Unit 1 introduces the subject of risk management. It illustrates the structure of the module and presents the main ideas and methods in the analysis of financial investment. You will be shown how to measure the expected return and the variance of a financial portfolio, and why it is important to look at the covariances between securities. Finally, you will see how to compute the portfolio possibilities set under risk, and how to modify it when short sales are allowed and when investors are able to lend or borrow at a safe rate.

Learning outcomes

When you have completed your study of this unit and its readings, you will be able to:

- discuss what is meant by risk management
- explain the main forms of risk faced by financial and non-financial institutions
- outline the nature of a financial portfolio
- define the characteristics of the mean–variance approach
- compute the expected return and the variance of the returns on a financial security
- compute the expected return and the variance of portfolios of financial securities
- explain the covariance between the returns on two financial securities and the correlation coefficient – how they are related and why they are important
- explain the opportunity set under risk, and define efficient and inefficient portfolios
- discuss how short sales can expand the opportunity set of investors
- assess how the opportunity set can be modified by the possibility of riskless lending and borrowing.

Reading for Unit 1

Crouhy M, D Galai and R Mark (2014) Appendix to Chapter 1 'Typology of Risk Exposures' of *The Essentials of Risk Management*.

Persaud A (2008) 'Regulation, valuation and systemic liquidity'.

Elton EJ, MJ Gruber, SJ Brown and WN Goetzmann (2014) Sections of Chapters 1–5 of *Modern Portfolio Theory and Investment Analysis*.

1.1 Introduction to Portfolio Analysis

The first unit of this module gives a general introduction to Risk Management, and lays the foundation for the analysis of investment that will be pursued in all later units of the module. You will look at the main forms of risk that investors have to face, both in financial and in non-financial institutions. The various types of risk will usually require different risk management strategies, and this will be the main concern of this module. In the present unit, you will look mainly at the basic principles behind the most fundamental risk management tool: the analysis of portfolios of risky assets, or *portfolio analysis*.

But what exactly is meant by 'portfolio analysis'?

In general, a collection of assets (real or financial) is called a *portfolio*. If you look at your own personal wealth, for instance, you will notice that it is ordinarily held in the form of a portfolio of assets. You might hold cash, bank deposits, bonds, insurance policies, and might also own durable commodities, cars, houses, *etc*. In general, portfolios can be composed of both real and financial assets.

The key idea in portfolio analysis is that, when investors are trying to establish the value of their wealth, they must consider their assets *as a whole*. Thus, in principle, investors should not consider the value to themselves of any of their real assets independently of the value of their other real assets or of their financial securities.

The reason for that can be seen by considering the following highly simplified example. Suppose that an investor holds her wealth entirely in the form of long-term bonds and a house. Let us assume that the market value of houses in the economy is affected by the level of interest rates: you can assume, in particular, that an increase in interest rates *reduces* the market value of houses, due to the increase in the burden of mortgage repayments and a reduced demand for houses in the economy.

Suppose now that there is an increase in the rates of interest in the economy. This would lead to a fall in the market value of the house. However, this is also likely to result in a reduction in the market price of bonds (you will see this in more detail in Unit 3 of the module, on the management of bond portfolios). Thus, the possibility that the investor might realise capital gains or losses on her bonds is associated with the possibility of capital gains or losses on the market value of her house. In other words, the investment as a whole in the portfolio happens to be very sensitive to fluctuations in interest rates. The reason for this is that the value of both bonds and house prices tends to be sensitive to changes in the rates of interest, and their returns tend to vary in the same direction (when the price of bonds increases the same is usually true of house prices, and *vice versa*).

The previous extremely simple example shows that, when analysing a financial investment, you must always consider the portfolio of assets *as a whole*, rather than single assets in isolation. You cannot just consider the expected returns on each asset, but must also be concerned with how those returns *vary together*. The co-movements of the returns are a crucial feature of the *risk* of the portfolio. The focus of this unit, as well as of Units 2 and 3 of the module, is on *portfolios*, in which you look at *collections of assets*. You will study how you can measure the risk and return of a portfolio, and how you can select portfolios of bonds and stocks.

1.2 Risks Faced by Financial and Non-financial Institutions

The first step in risk management is the identification of the sources of risk by which institutions can be affected, and the analysis of how risk can affect their profits. In general, it is important to distinguish between financial and non-financial institutions, since there could be important differences in the way they relate to risk. Regarding financial institutions, these often act as intermediaries amongst investors with a different risk exposure. Their role is therefore to offer opportunities for diversification and the hedging of risks. Conversely, they may be asked to act so as to increase the risk profile faced by investors, if these investors want to speculate.

By contrast, non-financial institutions face risks in the course of their normal business activity. These could be related to uncertainties in output markets, exchange rate fluctuations, *etc*. Non-financial institutions may seek to limit the effects of risks on their profits.

The risks faced by financial institutions may be classified as follows:

1. *Market risk*: this is due to changes in prices and rates of interest in financial markets. Both the assets and the liabilities of financial institutions consist of financial securities. Hence, the value of their net position can be affected by changes in equity prices, interest rates, exchange rates, *etc*.
2. *Credit risk*: this risk consists of changes in the credit quality of the counterparties, which affect the financial institution's position by altering the credit quality of its balance sheet.
3. *Liquidity risk*: this is the risk associated with the institution's ability to raise the necessary funds to meet its needs for liquidity and/or to carry out the desired financial transactions.
4. *Operational risk*: this is the risk associated with the ordinary activities of the institutions, and can be associated, for instance, with a breakdown in software systems, management failures, fraud, human error, *etc*.
5. *Legal and regulatory risk*: this risk is associated with the possibility of changes in the regulatory framework or in the tax laws. Sometimes it can also involve the difficulty of enforcing a financial contract or engaging in a transaction.
6. *Systemic risk*: this risk could be associated with a systemic collapse of the banking industry at regional, national or international level ('domino effect'). This could be due to the occurrence of bank failures,

whose consequences rapidly spread throughout the system and generate a collapse in the level of confidence towards the banking system as a whole.

Similarly, the risks faced by non-financial institutions can be classified as:

1. *Business risk*: this is the risk associated with the ordinary operations of the institutions: fluctuations in demand and supply, price changes, competitive pressures, *etc*.
2. *Operational risk*: this includes the risk associated with technical progress, such as the need to replace production processes due to obsolescence.
3. *Market risk*: this includes the risk to the firm's profits due to changes in inflation, interest rates, fluctuations of exchange rates, *etc*.
4. *Credit risk*: this includes changes in the institution's own credit rating or in the credit rating of its clients, which could affect the firm's cost of obtaining funds for its investment projects.

In addition, both financial and non-financial institutions are increasingly facing reputation risk. The awareness of reputation risk is becoming more widespread following a number of accounting scandals involving large corporations. Also, the increasing complexity and use of structured financial products have led some to question the legality of related transactions, and whether such transactions are suitable for all financial institutions. The major question remains how can reputation risk be measured, and should institutions maintain capital in recognition of reputation risk?

Reading 1.1

You can find a discussion of the typology of risk exposures for financial and non-financial institutions in Crouhy *et al*, Chapter 1, Appendix 1.1, 'Typology of risk exposures', pp. 23–43. Please stop now and read this section. You may want to read the remaining sections of the chapter, but that is not essential reading.

Crouhy *et al* (2014) *The Essentials of Risk Management*, Appendix to Chapter 1, 'Risk Management – A Helicopter View'.

The next reading is a short article by Avinash Persaud, which provides useful explanation and intuition on liquidity risk. The author also considers the various liquidity risk positions of different types of financial institution. He explains what happens when different types of financial institution start to exhibit the same behaviour as occurred in the credit crunch of 2007/08. The article covers market risk, credit risk and systemic liquidity risk, risk traders and risk absorbers, and factors that contributed to the 2007/08 credit crunch. It also touches briefly on a number of themes you will study in other units in this module. You should read the whole article, but do not give too much attention to the detail of accounting rules, supervision and regulation.

Reading 1.2

Please now read the article 'Regulation, valuation and systemic liquidity' by Avinash Persaud.

Be sure your notes cover the main issues cited above.

Persaud (2008) 'Regulation, valuation and systemic liquidity' Banque de France *Financial Stability Review*.

1.3 Financial Securities and Financial Markets

The main instruments traded in financial markets are known as *financial securities*. The various types of financial securities involve different risks, and play a different role in the investors' portfolios. They may also require different management techniques. By convention, financial securities can be distinguished into:

- money market securities
- capital market securities
- derivative securities.

You will find a description of the main types of financial securities in Chapter 2 of Elton *et al* (2014). Money market securities are short-term financial instruments. When they are issued, their maturity is at most one year. The main money market securities are treasury bills, repurchase agreements (or Repos), certificates of deposit, bankers' acceptances, and commercial paper.

Capital market securities are those financial instruments whose maturity is greater than one year when they are issued. They include treasury bonds, corporate bonds, common stock (or equity), and preferred stock.

Derivative securities are so called because their value derives from the value of an underlying asset or security (*eg* a stock). Derivatives can also be written so that their value depends on a commodity, such as cocoa or oil. The most important derivative securities are *futures* and *options*.

In this module, you will study in detail the risk characteristics of the various types of financial securities, and you will examine the most appropriate risk management strategies for each category of security.

Reading 1.3

Please now read Chapter 2 of Elton *et al* (2014). This chapter contains a description of both money market and capital market securities, which form the main instruments of financial portfolios.

It is important that you familiarise yourself with the definitions of all these instruments, so make sure your notes cover these.

Elton *et al* (2014) *Modern Portfolio Theory and Investment Analysis*, Chapter 2 'Financial Securities'.

Chapter 3 of Elton *et al* deals with the mechanics of financial markets. You will not need a detailed knowledge of the working of financial markets for this module on *Risk Management*. Hence, you do not need to study this chapter in great detail. There are, however, two topics which can be quite important in portfolio analysis and risk management. The first one is *short sales*. These consist of the sale of securities which investors do not own. These operations are usually intermediated by brokerage firms, so that the investor does not generally know the identity of the actual owner of the securities that are borrowed. Short sales are explained on pages 25–26 of Elton *et al* (2014).

The second important topic is represented by the *margins* on levered investments. These consist of the deposit, or the cash amount, which is paid when purchasing securities (the remaining amount can be borrowed). They are subject to two types of regulations. These are the regulations that control

- the amount which investors can borrow when purchasing securities (initial margin requirements) and
- the extent to which the value of the margin can fall relative to the value of the assets before action must be taken to restore the margin at the appropriate level (*maintenance margin requirements*).

Margins are also important for futures and options investment. They are briefly explained by Elton and his colleagues on pages 27–30.

Reading 1.4

Please now skim quickly through Chapter 3 of Elton *et al* (2014). You should, however, pay close attention to the operations of short sales and margins, both of which are described in Chapter 3, on pages 25–26 and 27–30, respectively.

Elton *et al* (2014) *Modern Portfolio Theory and Investment Analysis*, Chapter 3 'Financial Markets'.

1.4 The Mean–Variance Approach

The mean–variance approach to portfolio analysis is originally due to Harry Markowitz, who developed it in the 1950s. Any investment in financial securities is associated with a fundamental uncertainty about its outcome. It is generally impossible to predict with certainty the actual return from an investment. What you can do is to characterise the uncertainty about the likely outcomes in terms of a *probability distribution*, which summarises your degree of belief about the likelihood of the possible returns. This probability distribution could be based on the past historical performance of the securities, possibly modified to reflect the investors' knowledge of the current market conditions.

On the basis of the probability distribution of returns, you can compute the *mean return*, or *expected return*, on the securities, which is a measure of the 'centre' of the distribution. You could also compute the *variance*: this is a measure of the spread, or dispersion, of the securities about their mean value. The square root of the variance, which is called the *standard deviation*, is also a widely used measure of dispersion.

Box 1.1

The main practical difference between the standard deviation and the variance is that the former is expressed in the same unit of measurement as the returns, whereas the variance would be measured in 'returns squared'. In computing the variance we take the deviations of each value from the mean, square them, and then take an average.

According to the mean–variance approach to portfolio analysis, all an investor needs to know about a portfolio of securities are the mean and the variance (or, equivalently, the standard deviation). Investors will have preferences over the mean and the variance of portfolios: they prefer a greater expected return to less, and (since they are assumed to be risk averse) less variance to more. Their utility function is thus an increasing function of the expected return, and a decreasing function of the variance of their investment. In principle, they may be happy to accept a greater risk (*ie* a larger variance for their investment) if this is associated with a sufficiently large increase in the expected return. The portfolio selection problem therefore goes as follows.

Box 1.2

Investors compute the mean and the variance of all possible investment portfolios, and then select that portfolio which maximises their utility in terms of the mean–variance combination it offers.

All the relevant information for investors is thus summarised in the mean return from the investment and its variance. A more detailed knowledge of the distribution of returns would be irrelevant. This is clearly a very powerful simplification of the original problem of choice under uncertainty, since it enables investors to concentrate on just these two summary measures of the distribution of returns. Although this approach is not completely general (there could be instances, for example, in which investors are concerned with the possible *asymmetry* of the distribution, and this is not captured by the mean and the variance alone), in practice this simplification is usually regarded as satisfactory for a large class of investment problems, and has proven itself to be very useful in many actual applications.

🕮 Reading 1.5

Please now read Chapter 4 from Elton *et al* (2014), pages 42–62.

Elton *et al* (2014) ***Modern Portfolio Theory and Investment Analysis,*** Chapter 4, 'The Characteristics of the Opportunity Set under Risk'.

✍ As you read, make sure your notes enable you to answer the following questions:

- How can you determine the expected (or also mean or average) return of an asset given the probability distribution of its returns?
- How can you compute the variance and the standard deviation of the returns of an asset given the probability distribution of its returns?
- How can you compute the expected return and the variance of a combination of assets?
- What is the covariance between two assets? How can you interpret it? What does it mean if the covariance between two assets is positive or negative?
- What happens when you form large portfolios? What are the main implications of diversification? What is the role of covariances?

1.4.1 Mean and variance of one asset

Suppose we consider N financial assets, where the return on the i-th asset can take the M values $R_{i1}, R_{i2}, \ldots, R_{iM}$ with probabilities $P_{i1}, P_{i2}, \ldots, P_{iM}$. The mean return, or expected return, on the i-th asset is given by

$$\overline{R}_i = P_{i1}R_{i1} + P_{i2}R_{i2} + \ldots + P_{iM}R_{iM}$$

This value should be computed for each asset $i = 1,\ 2,\ ...,\ N$. The formula for the expected return can be expressed in a more compact form by using the summation symbol, $\sum$:

$$\overline{R}_i = \sum_{j=1}^{M} P_{ij}R_{ij} \qquad i = 1,\ 2,\ ...,\ N \qquad (1.1)$$

The variance of the i-th asset is given by

$$\sigma_i^2 = \sum_{j=1}^{M} P_{ij}(R_{ij} - \overline{R}_i)^2 \qquad (1.2)$$

The standard deviation, defined as the square root of the variance, is denoted by σ_i. The mean return is a measure of the *central tendency* of the distribution, and the variance and standard deviation are measures of the *spread around its centre*.

The summation symbol $\sum$ is a very useful tool to represent a summation in a compact fashion. Suppose you want to add together n terms, a_1, a_2, …, a_i, …, a_n:

$$a_1 + a_2 + ... + a_i + ... + a_n$$

By using the summation symbol, you can simply write this sum as:

$$\sum_{i=1}^{n} a_i$$

1.4.2 Mean, variance and covariance of a portfolio

If you hold a combination, or portfolio, of assets you could similarly compute the expected return and the variance of the whole portfolio. Suppose assets 1, 2, ..., N are held in the proportions $X_1, X_2,\ \ldots, X_N$ respectively: then the expected return on the portfolio, denoted by $\overline{R}_p$, is given by

$$\overline{R}_p = E(R_p) = \sum_{i=1}^{N} X_i \overline{R}_i \qquad (1.3)$$

where $X_1 + X_2 + \ \ldots\ + X_N = 1$. When computing the variance, we also have to concern ourselves with how the asset returns *vary together*, or *covary*. If the returns tend to move in opposite directions (that is, when the returns on some of the assets are high, the returns on some others are usually low, and *vice versa*), then this produces the effect of reducing the overall variability of the portfolio. By contrast, if returns tend all to move in the same direction, then the variability of the portfolio is increased. The statistical measure of how two assets move together is given by their *covariance*: the covariance between assets 1 and 2 is denoted by σ_{12}, and is defined as

$$\sigma_{12} = \sum_{j=1}^{M} P_{12,j}(R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2) \tag{1.4}$$

where $P_{12,j}$ denotes the *joint probability* that the return on the first asset is equal to R_{1j} and the return on the second asset is equal to R_{2j}. You could similarly define the covariance between two generic assets *i* and *k*, to be denoted by σ_{ik}.

If σ12 is positive, when the return on the first asset is greater than the mean value $\overline{R}_1$ (that is, $R_{1j} - \overline{R}_1 > 0$), then the return on the second asset is also, on average, greater than its mean value $\overline{R}_2$ (that is, $R_{2j} - \overline{R}_2 > 0$).

Conversely, when $R_{1j} - \overline{R}_1 < 0$, then we also have that, on average, $R_{2j} - \overline{R}_2 < 0$. Thus, the returns on the two assets tend to move, on average, in the same direction.

By contrast, if σ12 is negative, the returns on assets 1 and 2 tend to move in opposite directions and to offset each other: when the return on the first asset is greater than its mean value $\overline{R}_1$ (that is, $R_{1j} - \overline{R}_1 > 0$), then the return on the second asset is, on average, less than its mean value $\overline{R}_2$ (that is, $R_{2j} - \overline{R}_2 < 0$), and *vice versa*. The fact that the returns on the assets move in opposite directions reduces the variability of the portfolio. In fact, the portfolio tends to be insulated from shocks to the returns on the assets which form it.

In general, the variance of a portfolio of assets is given by

$$\sigma_P^2 = Var(R_P) = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq i}}^{N} X_i X_k \sigma_{ik} \tag{1.5}$$

and its standard deviation is $\sigma_P = SD(R_P) = \sqrt{\sigma_P^2}$. A measure of the association between two assets, which is always in the range between –1 and +1, is the *correlation coefficient*. The correlation coefficient between assets 1 and 2, denoted as ρ_{12}, is defined as

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \tag{6}$$

where $\sigma_1 = \sqrt{\sigma_1^2}$ and $\sigma_2 = \sqrt{\sigma_2^2}$ are the standard deviations of assets 1 and 2 respectively. The correlation coefficient always has the same sign as the covariance. Thus, if $\rho_{12} > 0$ then the returns on assets 1 and 2 will tend to move, on average, in the same direction. If $\rho_{12} < 0$, they will tend to move in opposite directions. Furthermore, it is always true that $-1 \leq \rho_{12} \leq 1$.

Exercise 1.1 and 1.2

Now please solve problems 1 and 2 on pages 62–63 of Elton *et al* (2014). These problems require you to compute mean returns, standard deviations, covariances and correlation coefficients for assets and for portfolios of assets. It is important that you familiarise yourselves with these computations. This is also a good way to make sure that you have really understood the theory. Answers are provided.

1.4.3 Variance of a portfolio: diversification and risk

What happens to the variance of a portfolio as the number of assets gets large? Look again at the formula for the variance, σ_P^2, and assume that all the N assets are held in equal proportions: this means that $X_i = 1/N$, for all i. The variance becomes

$$\sigma_P^2 = \sum_{i=1}^{N} \frac{1}{N^2}\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{k=1 \\ k\neq i}}^{N} \frac{1}{N^2}\sigma_{ik} \qquad (1.7)$$

We could try to write the above formula in terms of the *average variance* for the assets and of their *average covariance*. These can be defined as

$$\bar{\sigma}_i^2 = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 \qquad\qquad \bar{\sigma}_{ik} = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{\substack{k=1 \\ k\neq i}}^{N}\sigma_{ik} \qquad (1.8)$$

(Note that there are $N(N-1)$ elements inside the double summation symbol in the formula for $\bar{\sigma}_{ik}$, since each of the N assets can be combined with any of the remaining $N-1$ assets to compute the covariance.) The variance of the portfolio can therefore be written as

$$\sigma_P^2 = \frac{1}{N}\bar{\sigma}_i^2 + \frac{N-1}{N}\bar{\sigma}_{ik} \qquad (1.9)$$

We can now see what happens to σ_P^2 as the number of assets gets large (*ie* as $N \to \infty$). The coefficient $1/N$ on $\bar{\sigma}_i^2$ tends to become negligible, whereas the coefficient $(N-1)/N$ on $\bar{\sigma}_{ik}$ tends to unity (the numerator is about equal to the denominator, when N is a large number). Thus, as the number N of assets in a portfolio increases, the importance of the variances of the individual assets tends to vanish (since $1/N \to 0$), and the variance of the portfolio converges to the *average covariance* between the assets (since $(N-1)/N \to 1$).

Formally, we write

$$\sigma_P^2 \to \bar{\sigma}_{ik} \qquad \text{as} \qquad N \to \infty$$

In the jargon of portfolio analysis, individual risk is completely *diversified away*: individual variances play no role whatsoever in a large portfolio. Elton *et al* (2014) provide an example of this result in their Table 4.8, page 57.

1.5 The Opportunity Set under Risk – Efficient Portfolios

The previous section has illustrated how you can summarise the uncertain returns on an asset in terms of its mean and variance (or, equivalently, the mean and the standard deviation). You have also seen how to compute the mean and the variance of the returns on a portfolio that comprises a collection of assets. The fundamental important result that was obtained in the previous section is that the properties of a portfolio can be very different from those of the underlying assets, when considered individually. In particular, in large portfolios the variability of individual assets is completely diversified away, and the risk of the portfolio only depends on the *average covariance* between the pairs of assets.

How can you use these results to construct 'good' portfolios? Ideally, you would like to form portfolios with a large expected return and a small variance.

The first step towards forming portfolios of financial assets is to analyse the combinations of a limited number of risky assets, in order to examine how the properties of the combinations of assets are related to the underlying securities. You will then be in a position to evaluate more complex portfolios, involving a large number of assets.

We have already established the importance of the covariances and correlation coefficients when looking at the variability of the returns on a portfolio. In the present section I will ask you to consider simple portfolios obtained by combining *two* assets only, and to examine how the properties of the resulting combinations critically depend on the correlation coefficient between the underlying assets.

Reading 1.6

The material covered in this section is presented in the first sections of Chapter 5 of Elton *et al* (2014), pages 65–74. Please read those pages now. You should make sure you can follow all the algebraic steps of the presentation.

Elton *et al* (2014) *Modern Portfolio Theory and Investment Analysis,* Chapter 5 'Delineating Efficient Portfolios'.

Suppose you consider two assets, A and B, with expected returns $\overline{R}_A = E(R_A)$ and $\overline{R}_B = E(R_B)$ respectively, with variances σ_A^2 and σ_B^2, and with covariance σ_{AB}. If you form a portfolio P in which a proportion X_A is invested in asset A and a proportion $X_B = (1 - X_A)$ is invested in asset B, the expected return of the portfolio P is

$$\overline{R}_P = X_A \overline{R}_A + (1 - X_A)\overline{R}_B \qquad (1.10)$$

and the variance is

$$\sigma_P^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\sigma_{AB} \qquad (1.11)$$

The standard deviation can be written, using the equality $\rho_{AB} = \sigma_{AB} / \sigma_A \sigma_B$, as

$$\sigma_P = [X_A^2\sigma_A^2 + (1 - X_A)^2\sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B]^{1/2} \qquad (1.12)$$

These are extremely useful results. The expected return from the portfolio depends on the expected returns on the underlying assets, $\bar{R}_A$ and $\bar{R}_B$, and on the relative proportions in which they are held, X_A and $(1 - X_A)$. By contrast, the risk associated with the portfolio P is not only related to the variability of the underlying risky assets, as measured by their variances σ_A^2 and σ_B^2, and to their relative proportions, X_A and $(1 - X_A)$, but also to their correlation coefficient, ρ_{AB}. In particular, if the correlation coefficient between A and B is negative, the risk of the portfolio will be reduced. This should be an intuitive result: if $\rho_{AB} < 0$, then the returns on A and B will tend to move in opposite directions, and will partially offset each other. In other words, when the return on one of the assets is high, the return on the other asset will on average be low. As a result, the portfolio formed of A and B is *less variable* than the underlying individual assets, because the movements in the returns tend to cancel each other out.

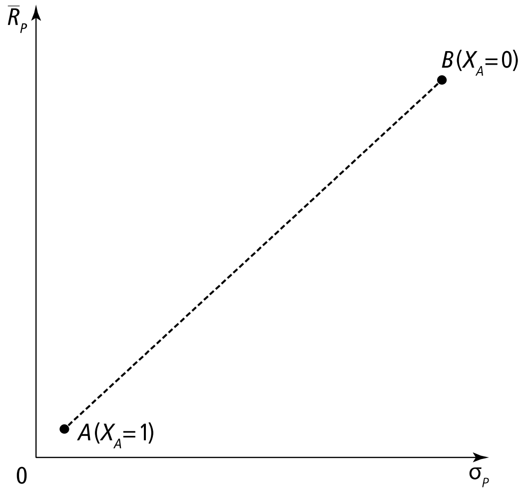
The ability of investors to reduce the variability of their position by diversifying therefore depends on the correlation coefficients between the assets being relatively low, or even negative. If the correlation coefficient is close to +1, then the returns on the assets will always tend to vary in the same direction, and the risk-reduction advantages from diversification are negligible.

Elton and his colleagues distinguish four cases, depending on whether the correlation coefficient is equal to +1, –1, 0 or 0.5. In the first case, the standard deviation of the portfolio P is:

$$\sigma_P = X_A\sigma_A + (1 - X_A)\sigma_B \qquad (1.13)$$

(Elton *et al,* 2014: p. 67). When $\rho_{AB} = +1$, there is no risk reduction from diversification: the standard deviation of the portfolio is simply the weighted average of the standard deviations of A and B. Figure 1.1 below illustrates this case. At point A, the portfolio only includes asset A: that is, $X_A = 1$. At point B, the portfolio only includes asset B: $X_B = 1$, and therefore $X_A = 0$. As X_A decreases from 1 to 0, the point on the graph representing the portfolio moves along the straight line from point A to point B.

Figure 1.1



When $\rho = -1$, by contrast, the variance of the portfolio is

$$\sigma_P^2 = [X_A\sigma_A - (1 - X_A)\sigma_B]^2 \tag{1.14}$$

and the standard deviation of the portfolio is:

$$\sigma_P = \begin{cases} X_A\sigma_A - (1 - X_A)\sigma_B & \text{if } X_A\sigma_A - (1 - X_A)\sigma_B \geq 0 \\ (1 - X_A)\sigma_B - X_A\sigma_A & \text{if } X_A\sigma_A - (1 - X_A)\sigma_B \leq 0 \end{cases} \tag{1.15}$$

(Elton *et al*, 2014: p. 69). The reason for having two separate analytical expressions for the standard deviation is that we have to make sure that σ_P is defined to be non-negative, when we take the square root of the variance.

The important result is that we can now *reduce the variability of our investment by diversifying*. In particular, we can form *a portfolio with zero risk*. This is obtained by setting the standard deviation equal to zero: $\sigma_P = 0$. This yields

$$X_A = \frac{\sigma_B}{\sigma_A + \sigma_B} \tag{1.16}$$

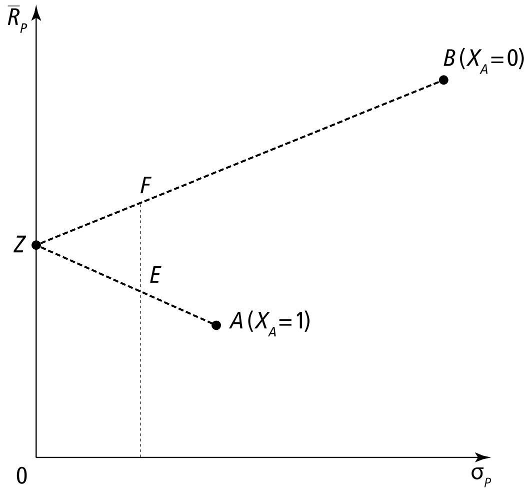
and thus also

$$X_B = 1 - X_A = \frac{\sigma_A}{\sigma_A + \sigma_B} \tag{1.17}$$

When the assets *A* and *B* are held in exactly the above proportions, the portfolio *P* will have zero variance: hence, the return on *P* will always be equal to the expected return, $\bar{R}_P$. In this case, the advantages from diversification are potentially very large, since you will be able to completely eliminate the risk from your portfolio of (risky) assets.

This situation is illustrated by Figure 1.2.

Figure 1.2



At point A, the portfolio only includes asset A: that is, $X_A = 1$. At point B, the portfolio only includes asset B: $X_B = 1$, and therefore $X_A = 0$. As X_A decreases from 1 to 0, the point on the graph representing the portfolio moves along the broken straight line from A to B. When $X_A = \sigma_B/(\sigma_A + \sigma_B)$, the portfolio is described by point Z: this is the *zero-variance portfolio*. It lies on the vertical axis, since its variance (and hence also its standard deviation) is equal to zero.

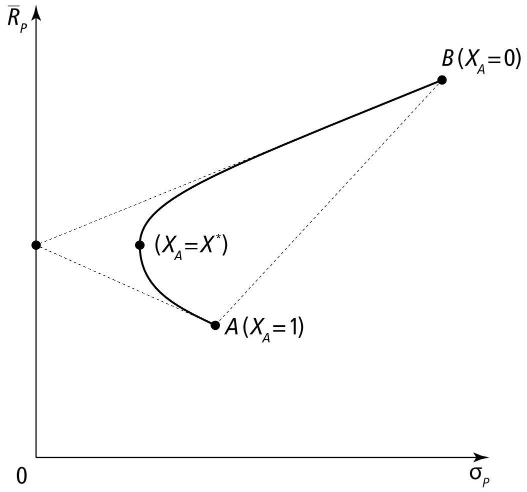
It is important to note that all portfolios on the lower segment of the line from A to Z have the same standard deviation, but lower expected return, compared to portfolios that lie on the upper segment of the line, from Z to B. For instance, the portfolio represented by point E has the same risk (in terms of the standard deviation), but a lower expected return, compared to the portfolio described by point F. If you are a rational, risk-averse investor, you will never choose to hold portfolios on the line from A to Z: these portfolios are *inefficient*. The reason is that these portfolios are dominated by portfolios on the upper segment, which offer a higher expected return for a given standard deviation. The only efficient portfolios are thus on the line from Z to B: they form the *set of efficient portfolios*.

In practice, the correlation coefficient will usually be a number between –1 and +1. When $\rho = 0$, for instance, the standard deviation of the portfolio is

$$\sigma_P = [X_A^2\sigma_A^2 + (1 - X_A)^2\sigma_B^2]^{1/2} \tag{1.18}$$

(see also Elton *et al*, 2014: p. 71). The set of portfolios formed by combining assets A and B is now described not by a straight line, but by a curve on the $(\sigma_P, \bar{R}_P)$ plane (Figure 1.3).

Figure 1.3



Diversification can still reduce the risk of the portfolio, but we can no longer construct a zero-variance portfolio (we can only do this when $\rho = -1$). However, we can still find a *minimum-variance portfolio*. This is obtained by selecting that value of X_A for which the variance (or, equivalently, the standard deviation) is smallest. This problem can be solved by taking the derivative of σ_P^2 (or of σ_P) with respect to X_A and setting it equal to 0. This is shown in detail by Elton *et al* (2014: pp. 71–74) for the general case in which $-1 < \rho < 1$; when $\rho = 0$, the critical value of X_A for which the variance σ_P^2 is minimised is

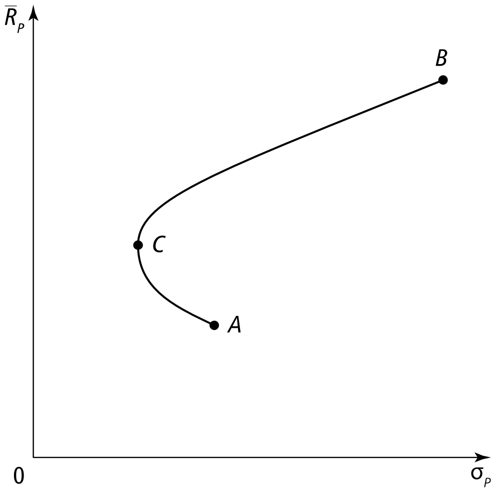
$$X_A = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2} \tag{1.19}$$

This is shown as point *C* in Figure 1.4 below.

The portfolios that lie on the line from *A* to *C* are *inefficient*, since for each one of them we can find an alternative portfolio, on the line from *C* to *B*, which has the same standard deviation but a higher expected return (by the same reasoning we used for Figure 1.2). The *set of efficient portfolios* is now formed of the line from point *C* to point *B*.

Thus, given two risky assets *A* and *B*, we can represent the set of all their possible portfolios as a curve connecting the two assets on the standard deviation–expected return plane. This has been illustrated for $\rho = +1$, $\rho = -1$ and $\rho = 0$. When $\rho = +1$, the investment possibilities are a straight line connecting the assets *A* and *B*. When $\rho = -1$, the investment possibilities are a broken straight line, which intersects the vertical axis at the zero-variance portfolio. These are the two extreme cases: for intermediate values of the correlation coefficient (that is, when ρ is strictly greater than –1 and strictly less than +1) the portfolio possibilities will be described by a concave curve connecting the assets (as in Figures 1.3 and 1.4).

Figure 1.4



Exercise 1.3

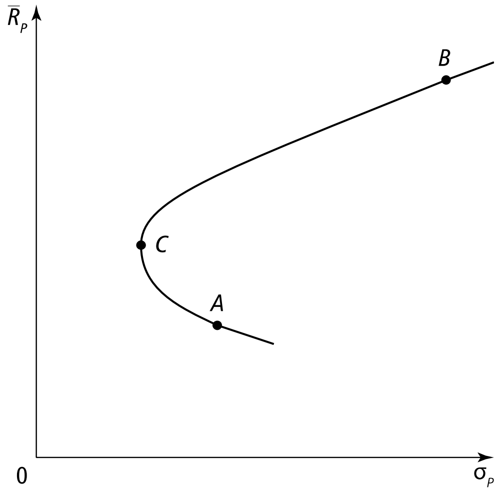
Please now solve problem 1 part A on page 92 of Elton *et al* (2014). An answer is provided.

1.6 Short Sales and Riskless Lending and Borrowing

There are two important extensions to the analysis carried out in the previous section. First, you have to consider what happens when you are allowed to *sell short* some of your assets – that is, you can sell assets that you do not yet own. Second, you have to describe your portfolio possibilities when you can lend at a riskless rate (for instance, if you can purchase very short-term government bills), or borrow at the riskless rate.

Let us start by considering the possibility of short-selling assets (or selling assets short). If you have two risky assets, *A* and *B*, the portfolio possibility set can be described by a curve connecting them, as in Figure 1.5. At point A, 100% of wealth is invested in asset *A*, and at point *B*, 100% of wealth is invested in asset *B*. However, if short sales are allowed, investors can *sell short* asset *A*, and increase their holdings of asset *B*. If they do that, they will be able to move *to the right of point B* on the curve in the figure. In fact, *more than 100% of their original wealth* is invested in asset *B*. They are able to invest an amount greater than their wealth, because they have sold short an asset (*A*) they do not own. Similarly, they could sell short asset *B*, increase their wealth, and invest everything in asset *A*. If they do that, they will hold a portfolio on the curve *to the right of point A*.

Figure 1.5



The total portfolio possibility set when short sales are allowed is thus the line going through *A* to *B*, but extending to the right of each point. From Figure 1.5, you can also see that the portfolios in the lower section of the curve from point *C* (including point *A*) are *inefficient*, since they are dominated by portfolios on the upper half of the curve. The latter portfolios have a higher expected return for any given standard deviation, and would therefore be chosen by rational investors who prefer more to less and who are risk averse. Hence, the *set of efficient portfolios when short sales are allowed* is the upper half of the curve from point *C*. Rational and risk-averse investors will select a portfolio from the efficient set.

Reading 1.7

Please now read the section of Chapter 5, Elton *et al* (2014) on pages 74–81. The authors first discuss the feasible shapes of the portfolio possibility curve in the absence of short sales, and then introduce short sales and examine how this process enhances the possibility set.

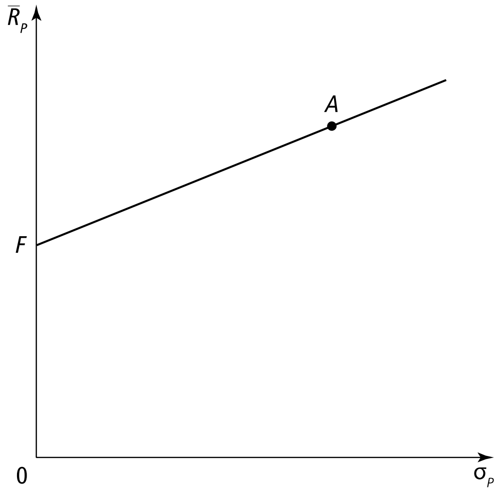
Elton *et al* (2014) *Modern Portfolio Theory and Investment Analysis,* Chapter 5 'Delineating Efficient Portfolios'.

As you read, please make sure you can answer the following questions.

- Why is the portfolio possibilities curve convex (with respect to the vertical axis)?
- Why do short sales increase the investment possibilities?
- What are the cash flows for an investor who has sold short financial securities?

Our next extension involves the possibility of investing in a *safe asset*. This can be regarded as a 'sure' investment – that is, an investment which delivers a promised return with certainty. Formally, the variance of its returns is equal to zero. The return on the asset is thus always equal to a constant value. As a consequence, this asset can be represented as a point *on the vertical axis*, on the standard deviation–expected return plane (point *F* in Figure 1.6).

Figure 1.6



What happens when we form a portfolio which includes the safe asset, F, together with a risky asset, A? In this case, the portfolio can be described as a point on the straight line connecting F to A (Figure 1.6). The expected return on the portfolio is given by

$$\overline{R}_P = (1 - X_A)R_F + X_A\overline{R}_A \tag{1.20}$$

where R_F is the return on the safe asset, $\overline{R}_A$ is the expected return on the risky asset, and X_A is the proportion of the risky asset held in the portfolio. Since the variance of the safe asset F is equal to zero, the variance of the return on the portfolio is simply:

$$\sigma_P^2 = X_A^2\sigma_A^2 \tag{1.21}$$

and therefore the standard deviation is:

$$\sigma_P = X_A\sigma_A \tag{1.22}$$

Investing in the safe asset F which gives a return R_F with certainty is equivalent to *lending at the safe rate of interest* R_F. If short sales are allowed, we could sell short the safe asset F: this would move us *to the right of point A* in Figure 1.6, and would be equivalent to *borrowing at the safe rate of interest* R_F.

Reading 1.8

Now read the rest of Chapter 5, pages 81–92 of Elton *et al* (2014), which first describes the analytics of lending and borrowing in the presence of a safe rate of return, and then illustrates a few examples of the efficient frontier. It is important that you read the text and the examples carefully.

Elton *et al* (2014) *Modern Portfolio Theory and Investment Analysis*, Chapter 5 'Delineating Efficient Portfolios'.

Exercise 1.4

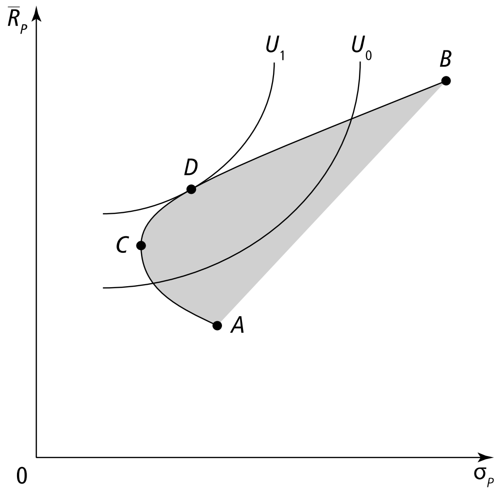
At the end of your reading, try to solve problem 5 on page 93 of Elton *et al* (2014). An answer is provided.

1.7 How to Compute the Efficient Set

The previous two sections discussed the shape of the portfolio possibilities of risky assets, with or without the presence of a risk-free (or safe) asset. When you can choose among many risky assets, and no pair of them is perfectly negatively correlated, then the set of all the portfolio possibilities is represented by a convex set, and the set of efficient portfolios by a concave line (the upper north-west frontier of the possibility set) on the (σ_P, $\overline{R}_P$) plane.

The exact choice of the investors will then be determined by their preferences, according to the mean–variance approach. An illustration of this is given in Figure 1.7.

Figure 1.7



The shaded area of the convex set *ACB* describes the portfolio possibilities, the line *CB* is the set of efficient portfolios, and U_0, U_1 are indifference curves for the investors.

An indifference curve measures risk-return combinations that yield the same level of overall utility to the investor. They are positively sloped, since investors require a higher expected return to compensate for a higher standard deviation. The curve U_1 corresponds to a higher utility level than U_0, since it corresponds to a higher expected return for any given standard deviation. The optimal choice for the investor is point *D*, where the indifference curve U_1 is tangent to the efficient frontier of the portfolio possibilities set.

But how can we find the efficient frontier *CB* of the set of portfolios, given the assets that are available?

The exact mathematical solution to this problem can be quite complicated to work out, and we shall not go into the details of the computations. In practice, investors use computer algorithms to find the solution to these problems. Elton and his co-authors (2014) outline a general approach to the problem in Chapter 6, and explain how to express the selection of the optimal portfolio in exact mathematical terms. This could be useful for two reasons. Firstly, you can understand how computer programs are constructed. Secondly, and more importantly for this module, expressing the problem in mathematical terms makes it apparent just how much information on the assets is required in order to find the efficient set. We need to know not just the expected values and variances of each asset, but also all the pairwise correlation coefficients. These informational requirements can be quite formidable, when the number of assets considered is even moderately large. Thus, the next units will look at some possible ways to simplify this problem.

Optional Reading 1.1

If you wish, you can now read Chapter 6 on 'Techniques for Calculating the Efficient Frontier' from Elton *et al* (2014), pages 95–106. This chapter is optional reading, and you can omit it without prejudice for the rest of the module.

1.8 Conclusion

This unit has described the fundamentals of the mean–variance approach. You have seen:

- how to compute the expected value and the variance of a financial portfolio, and
- how to compute the covariance between a pair of securities.

You have also examined:

- the benefits from diversification,

and have seen how:

- the variance of individual assets tends to have a negligible importance in large portfolios.

You have also studied:

- the shape of the opportunity set under risk.

The next units will present some ways to further simplify the mean–variance approach, and will put forward some operationally feasible methods for constructing financial portfolios.

References

Elton EJ, MJ Gruber, SJ Brown and WN Goetzmann (2014) *Modern Portfolio Theory and Investment Analysis.* 9th Edition. New York: John Wiley & Sons.

Crouhy M, D Galai and R Mark (2014) *The Essentials of Risk Management.* 2nd Edition. New York: McGraw Hill.

Persaud AD (2008) 'Regulation, valuation and systemic liquidity'. *Financial Stability Review*, No. 12, October, pp. 75–83, Banque de France.

Answers to Exercises

Exercise 1.1

Chapter 4, Exercise 1 (Elton et al*, (2014) pages 62–63)*

The calculations and plot are in Excel file (1997–2003 compatible) C323_U1_Elton_Chapter 4_Q1.xls

A

$E(R_1) = 12$ $\quad s_1 = 2.83$

$E(R_2) = 6$ $\quad s_2 = 1.41$

$E(R_3) = 14$ $\quad s_3 = 4.24$

$E(R_4) = 12$ $\quad s_4 = 3.27$

B

$s_{12} = –4$ $\quad s_{13} = 12$ $\quad s_{14} = 0$

$s_{23} = –6$ $\quad s_{24} = 0$ $\quad s_{34} = 0$

C

$E(Ra) = 9$ $\quad \sigma_a^2 = 0.5$

$E(Rb) = 13$ $\quad \sigma_b^2 = 12.5$

$E(Rc) = 12$ $\quad \sigma_c^2 = 4.6666 ... = 4.\overline{6}$

$E(Rd) = 10$ $\quad \sigma_d^2 = 2$

$E(Re) = 13$ $\quad \sigma_e^2 = 7.1\overline{6}$

$E(Rf) = 10.\overline{6}$ $\quad \sigma_f^2 = 3.\overline{5}$

$E(Rg) = 10.\overline{6}$ $\quad \sigma_g^2 = 2.07$

$E(Rh) = 12.\overline{6}$ $\quad \sigma_h^2 = 6.74$

$E(Ri) = 11$ $\quad \sigma_i^2 = 2.\overline{6}$

Exercise 1.2

Chapter 4, Exercise 2 (Elton et al*, (2014) page 63)*

See the Excel file C323_U1_Elton_Chapter 4_Q2.xls for details of the calculations.

A

Time	*RA*	*RB*	*RC*
2	0.0368	0.1051	0.0141
3	0.0038	0.0050	0.1492
4	–0.0653	0.0373	–0.0141
5	0.0135	0.0098	0.1084
6	0.0618	0.0339	0.0492
7	0.0212	–0.0145	0.1693

B	$E(R_A) = 0.0120$	$E(R_B) = 0.0295$	$E(R_C) = 0.0793$
C	$\sigma_A = SD(R_A) = 0.0392$	$\sigma_B = 0.0381$	$\sigma_C = 0.0680$
D	$\rho_{AB} = Corr(R_A, R_B) = 0.1406$	$\rho_{AC} = 0.2751$	$\rho_{BC} = -0.77435$

E

Portfolio	$E(R_P)$	σ_P
$\frac{1}{2}A + \frac{1}{2}B$	0.0207	0.0292
$\frac{1}{2}A + \frac{1}{2}C$	0.0457	0.0437
$\frac{1}{2}B + \frac{1}{2}C$	0.0544	0.0227
$\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$	0.0403	0.0247

Exercise 1.3

Chapter 5, Exercise 1 (Elton* et al*, (2014) page 92)

The calculations and graphs are included in Excel file C323_U1_Elton_Chapter 5_Q1.xls

A The correlation between returns for securities 1 and 2 is –1. In which case the minimum variance portfolio has zero variance, and the weights in such a portfolio are obtained from equations (1.16) and (1.17). The graph of expected portfolio return against portfolio standard deviation is obtained by varying the weights on security 1 from 0 to 1, in increments of 0.05, and calculating the expected returns and standard deviation. The zero variance portfolio is also included in the graph in the Excel file. You should obtain a graph similar to Figure 1.2.

(1) $X_1 = 1/3$ $X_2 = 2/3$ $E(R_P) = 8$ $\sigma_P = 0$

(4) **Assets 1 and 3:**

The correlation coefficient for the returns on assets 1 and 3 is +1. Therefore, the minimum variance portfolio is obtained by investing in the asset with the lower variance. The graph of portfolio expected return against standard deviation for various weights for asset 1 and asset 3 is again obtained by varying X_1 from 0 to 1 in increments of 0.05. It is similar to Figure 1.1.

$X_1 = 1$ $X_3 = 0$ $E(R_P) = 12$ $\sigma_P = 2.8284$

Assets 1 and 4:

The returns on assets 1 and 4 are independent, and therefore the covariance (and correlation coefficient) equals zero. The minimum variance portfolio is obtained by minimising the variance with respect to the weight of one of the assets; when the correlation coefficient is zero this leads to equation (1.19). In most cases you would expect to obtain a graph of expected portfolio return against portfolio standard deviation similar to Figure 1.4. However, the expected return for asset 1 equals the expected return for asset 4 (equals 12), so the graph of expected return against standard deviation for the portfolio (where the weights always sum to 1) is a horizontal straight line.

$X_1 = 0.5714 \quad X_4 = 0.4286 \quad E(R_P) = 12 \quad \sigma_P = 2.1381$

Assets 2 and 3:

The correlation coefficient between the returns on assets 2 and 3 is –1. It is possible to achieve a portfolio with zero variance, and the weights are obtained from equations (1.16) and (1.17). You should obtain a graph of portfolio expected return against standard deviation similar to Figure 1.2.

$X_2 = 0.75 \quad X_3 = 0.25 \quad E(R_P) = 8 \quad \sigma_P = 0$

Assets 2 and 4:

The correlation coefficient between returns on asset 2 and asset 4 is zero. The weights for the minimum variance portfolio are obtained from equation (1.19), and you should obtain a plot of portfolio expected return against standard deviation similar to Figure 1.4.

$X_2 = 0.8421 \quad X_4 = 0.1579 \quad E(R_P) = 6.9474 \quad \sigma_P = 1.2978$

Assets 3 and 4:

The correlation coefficient between the returns on asset 3 and asset 4 is also zero, and the same arguments and methods can be applied.

$X_3 = 0.3721 \quad X_4 = 0.6279 \quad E(R_P) = 12.7442 \quad \sigma_P = 2.5880$

Exercise 1.4

Chapter 5, Exercise 5 (Elton et al*, (2014) page 93)*

The Excel file C323_U1_Elton_Chapter 5_Q5.xls contains relevant calculations and plots. The plots of portfolio expected return against standard deviation are obtained by varying one weight from 0 to 1 in increments of 0.05.

$\rho = 1$:

When $\rho = 1$ the minimum variance portfolio is obtained by investing only in the asset with the lower variance. You should obtain a plot of portfolio expected return against standard deviation similar to Figure 1.1.

$X_1 = 0 \quad X_2 = 1 \quad \sigma_P = 2$

$\rho = 0$:

For $\rho = 0$ the weights in the minimum variance portfolio are obtained from equation (1.19), and you should obtain a plot of portfolio expected return against standard deviation similar to Figure 1.4.

$X_1 = 0.1379 \quad X_2 = 0.8621 \quad \sigma_P = 1.8570$

$\rho = -1$:

For $\rho = -1$ the minimum variance portfolio has zero variance; the weights are obtained from equations (1.16) and (1.17); and the graph of portfolio expected return against standard deviation should be similar to Figure 1.2.

$$X_1 = 0.2857 \qquad X_2 = 0.7143 \qquad \sigma_P = 0$$